

Exercise 13.1

Question 1:

Evaluate the Given limit: $\lim_{x \rightarrow 3} x + 3$

Answer

$$\lim_{x \rightarrow 3} x + 3 = 3 + 3 = 6$$

Question 2:

Evaluate the Given limit: $\lim_{x \rightarrow \pi} \left(x - \frac{22}{7} \right)$

Answer

$$\lim_{x \rightarrow \pi} \left(x - \frac{22}{7} \right) = \left(\pi - \frac{22}{7} \right)$$

Question 3:

Evaluate the Given limit: $\lim_{r \rightarrow 1} \pi r^2$

Answer

$$\lim_{r \rightarrow 1} \pi r^2 = \pi (1)^2 = \pi$$

Question 4:

Evaluate the Given limit: $\lim_{x \rightarrow 4} \frac{4x + 3}{x - 2}$

Answer

$$\lim_{x \rightarrow 4} \frac{4x + 3}{x - 2} = \frac{4(4) + 3}{4 - 2} = \frac{16 + 3}{2} = \frac{19}{2}$$

Question 5:

Evaluate the Given limit: $\lim_{x \rightarrow -1} \frac{x^{10} + x^5 + 1}{x - 1}$

Answer

$$\lim_{x \rightarrow -1} \frac{x^{10} + x^5 + 1}{x - 1} = \frac{(-1)^{10} + (-1)^5 + 1}{-1 - 1} = \frac{1 - 1 + 1}{-2} = -\frac{1}{2}$$

Question 6:

$$\lim_{x \rightarrow 0} \frac{(x+1)^5 - 1}{x}$$

Evaluate the Given limit:

Answer

$$\lim_{x \rightarrow 0} \frac{(x+1)^5 - 1}{x}$$

Put $x + 1 = y$ so that $y \rightarrow 1$ as $x \rightarrow 0$.

$$\begin{aligned} \text{Accordingly, } \lim_{x \rightarrow 0} \frac{(x+1)^5 - 1}{x} &= \lim_{y \rightarrow 1} \frac{y^5 - 1}{y - 1} \\ &= \lim_{y \rightarrow 1} \frac{y^5 - 1^5}{y - 1} \\ &= 5 \cdot 1^{5-1} \\ &= 5 \end{aligned}$$

$$\left[\lim_{x \rightarrow a} \frac{x^n - a^n}{x - a} = na^{n-1} \right]$$

$$\therefore \lim_{x \rightarrow 0} \frac{(x+1)^5 - 1}{x} = 5$$

Question 7:

$$\lim_{x \rightarrow 2} \frac{3x^2 - x - 10}{x^2 - 4}$$

Evaluate the Given limit:

Answer

$$\frac{0}{0}$$

At $x = 2$, the value of the given rational function takes the form $\frac{0}{0}$.

$$\begin{aligned} \therefore \lim_{x \rightarrow 2} \frac{3x^2 - x - 10}{x^2 - 4} &= \lim_{x \rightarrow 2} \frac{(x-2)(3x+5)}{(x-2)(x+2)} \\ &= \lim_{x \rightarrow 2} \frac{3x+5}{x+2} \\ &= \frac{3(2)+5}{2+2} \\ &= \frac{11}{4} \end{aligned}$$

Question 8:

Evaluate the Given limit: $\lim_{x \rightarrow 3} \frac{x^4 - 81}{2x^2 - 5x - 3}$

Answer

At $x = 3$, the value of the given rational function takes the form $\frac{0}{0}$.

$$\begin{aligned} \therefore \lim_{x \rightarrow 3} \frac{x^4 - 81}{2x^2 - 5x - 3} &= \lim_{x \rightarrow 3} \frac{(x-3)(x+3)(x^2+9)}{(x-3)(2x+1)} \\ &= \lim_{x \rightarrow 3} \frac{(x+3)(x^2+9)}{2x+1} \\ &= \frac{(3+3)(3^2+9)}{2(3)+1} \\ &= \frac{6 \times 18}{7} \\ &= \frac{108}{7} \end{aligned}$$

Question 9:

Evaluate the Given limit: $\lim_{x \rightarrow 0} \frac{ax+b}{cx+1}$

Answer

$$\lim_{x \rightarrow 0} \frac{ax+b}{cx+1} = \frac{a(0)+b}{c(0)+1} = b$$

Question 10:

$$\lim_{z \rightarrow 1} \frac{z^{\frac{1}{3}} - 1}{z^{\frac{1}{6}} - 1}$$

Evaluate the Given limit:

Answer

$$\lim_{z \rightarrow 1} \frac{z^{\frac{1}{3}} - 1}{z^{\frac{1}{6}} - 1}$$

At $z = 1$, the value of the given function takes the form $\frac{0}{0}$.

Put $z^{\frac{1}{6}} = x$ so that $z \rightarrow 1$ as $x \rightarrow 1$.

$$\text{Accordingly, } \lim_{z \rightarrow 1} \frac{z^{\frac{1}{3}} - 1}{z^{\frac{1}{6}} - 1} = \lim_{x \rightarrow 1} \frac{x^2 - 1}{x - 1}$$

$$= \lim_{x \rightarrow 1} \frac{x^2 - 1^2}{x - 1}$$

$$= 2 \cdot 1^{2-1}$$

$$= 2$$

$$\left[\lim_{x \rightarrow a} \frac{x^n - a^n}{x - a} = na^{n-1} \right]$$

$$\therefore \lim_{z \rightarrow 1} \frac{z^{\frac{1}{3}} - 1}{z^{\frac{1}{6}} - 1} = 2$$

Question 11:

$$\lim_{x \rightarrow 1} \frac{ax^2 + bx + c}{cx^2 + bx + a}, a + b + c \neq 0$$

Evaluate the Given limit:

Answer

$$\begin{aligned}\lim_{x \rightarrow 1} \frac{ax^2 + bx + c}{cx^2 + bx + a} &= \frac{a(1)^2 + b(1) + c}{c(1)^2 + b(1) + a} \\ &= \frac{a+b+c}{a+b+c} \\ &= 1 \quad [a+b+c \neq 0]\end{aligned}$$

Question 12:

$$\lim_{x \rightarrow -2} \frac{\frac{1}{x} + \frac{1}{2}}{x+2}$$

Evaluate the Given limit:

Answer

$$\lim_{x \rightarrow -2} \frac{\frac{1}{x} + \frac{1}{2}}{x+2}$$

$$\frac{0}{0}$$

At $x = -2$, the value of the given function takes the form $\frac{0}{0}$.

$$\begin{aligned}\text{Now, } \lim_{x \rightarrow -2} \frac{\frac{1}{x} + \frac{1}{2}}{x+2} &= \lim_{x \rightarrow -2} \frac{\left(\frac{2+x}{2x}\right)}{x+2} \\ &= \lim_{x \rightarrow -2} \frac{1}{2x} \\ &= \frac{1}{2(-2)} = \frac{-1}{4}\end{aligned}$$

Question 13:

$$\lim_{x \rightarrow 0} \frac{\sin ax}{bx}$$

Evaluate the Given limit:

Answer

$$\lim_{x \rightarrow 0} \frac{\sin ax}{bx}$$

$$\frac{0}{0}$$

At $x = 0$, the value of the given function takes the form $\frac{0}{0}$.

$$\begin{aligned}
 \text{Now, } \lim_{x \rightarrow 0} \frac{\sin ax}{bx} &= \lim_{x \rightarrow 0} \frac{\sin ax}{ax} \times \frac{ax}{bx} \\
 &= \lim_{x \rightarrow 0} \left(\frac{\sin ax}{ax} \right) \times \left(\frac{a}{b} \right) \\
 &= \frac{a}{b} \lim_{ax \rightarrow 0} \left(\frac{\sin ax}{ax} \right) && [x \rightarrow 0 \Rightarrow ax \rightarrow 0] \\
 &= \frac{a}{b} \times 1 && \left[\lim_{y \rightarrow 0} \frac{\sin y}{y} = 1 \right] \\
 &= \frac{a}{b}
 \end{aligned}$$

Question 14:

Evaluate the Given limit: $\lim_{x \rightarrow 0} \frac{\sin ax}{\sin bx}$, $a, b \neq 0$

Answer

$$\lim_{x \rightarrow 0} \frac{\sin ax}{\sin bx}, \quad a, b \neq 0$$

At $x = 0$, the value of the given function takes the form $\frac{0}{0}$.

$$\begin{aligned}
 \text{Now, } \lim_{x \rightarrow 0} \frac{\sin ax}{\sin bx} &= \lim_{x \rightarrow 0} \frac{\left(\frac{\sin ax}{ax} \right) \times ax}{\left(\frac{\sin bx}{bx} \right) \times bx} \\
 &= \left(\frac{a}{b} \right) \times \frac{\lim_{ax \rightarrow 0} \left(\frac{\sin ax}{ax} \right)}{\lim_{bx \rightarrow 0} \left(\frac{\sin bx}{bx} \right)} && \left[\begin{array}{l} x \rightarrow 0 \Rightarrow ax \rightarrow 0 \\ \text{and } x \rightarrow 0 \Rightarrow bx \rightarrow 0 \end{array} \right] \\
 &= \left(\frac{a}{b} \right) \times \frac{1}{1} && \left[\lim_{y \rightarrow 0} \frac{\sin y}{y} = 1 \right] \\
 &= \frac{a}{b}
 \end{aligned}$$

Question 15:

$$\lim_{x \rightarrow \pi} \frac{\sin(\pi - x)}{\pi(\pi - x)}$$

Evaluate the Given limit:

Answer

$$\lim_{x \rightarrow \pi} \frac{\sin(\pi - x)}{\pi(\pi - x)}$$

It is seen that $x \rightarrow \pi \Rightarrow (\pi - x) \rightarrow 0$

$$\begin{aligned} \therefore \lim_{x \rightarrow \pi} \frac{\sin(\pi - x)}{\pi(\pi - x)} &= \frac{1}{\pi} \lim_{(\pi - x) \rightarrow 0} \frac{\sin(\pi - x)}{(\pi - x)} \\ &= \frac{1}{\pi} \times 1 \quad \left[\lim_{y \rightarrow 0} \frac{\sin y}{y} = 1 \right] \\ &= \frac{1}{\pi} \end{aligned}$$

Question 16:

$$\lim_{x \rightarrow 0} \frac{\cos x}{\pi - x}$$

Evaluate the given limit:

Answer

$$\lim_{x \rightarrow 0} \frac{\cos x}{\pi - x} = \frac{\cos 0}{\pi - 0} = \frac{1}{\pi}$$

Question 17:

$$\lim_{x \rightarrow 0} \frac{\cos 2x - 1}{\cos x - 1}$$

Evaluate the Given limit:

Answer

$$\lim_{x \rightarrow 0} \frac{\cos 2x - 1}{\cos x - 1}$$

$\frac{0}{0}$.

At $x = 0$, the value of the given function takes the form $\frac{0}{0}$.

Now,

$$\lim_{x \rightarrow 0} \frac{\cos 2x - 1}{\cos x - 1} = \lim_{x \rightarrow 0} \frac{1 - 2 \sin^2 x - 1}{1 - 2 \sin^2 \frac{x}{2} - 1} \quad \left[\cos x = 1 - 2 \sin^2 \frac{x}{2} \right]$$

$$= \lim_{x \rightarrow 0} \frac{\sin^2 x}{\sin^2 \frac{x}{2}} = \lim_{x \rightarrow 0} \frac{\left(\frac{\sin^2 x}{x^2} \right) \times x^2}{\left(\frac{\sin^2 \frac{x}{2}}{\left(\frac{x}{2} \right)^2} \right) \times \frac{x^2}{4}}$$

$$= 4 \frac{\lim_{x \rightarrow 0} \left(\frac{\sin^2 x}{x^2} \right)}{\lim_{x \rightarrow 0} \left(\frac{\sin^2 \frac{x}{2}}{\left(\frac{x}{2} \right)^2} \right)}$$

$$= 4 \frac{\left(\lim_{x \rightarrow 0} \frac{\sin x}{x} \right)^2}{\left(\lim_{\frac{x}{2} \rightarrow 0} \frac{\sin \frac{x}{2}}{\frac{x}{2}} \right)^2} \quad \left[x \rightarrow 0 \Rightarrow \frac{x}{2} \rightarrow 0 \right]$$

$$= 4 \frac{1^2}{1^2} \quad \left[\lim_{y \rightarrow 0} \frac{\sin y}{y} = 1 \right]$$

$$= 4$$

Question 18:

Evaluate the Given limit: $\lim_{x \rightarrow 0} \frac{ax + x \cos x}{b \sin x}$

Answer

$$\lim_{x \rightarrow 0} \frac{ax + x \cos x}{b \sin x}$$

$\frac{0}{0}$.

At $x = 0$, the value of the given function takes the form $\frac{0}{0}$.

Now,

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{ax + x \cos x}{b \sin x} &= \frac{1}{b} \lim_{x \rightarrow 0} \frac{x(a + \cos x)}{\sin x} \\ &= \frac{1}{b} \lim_{x \rightarrow 0} \left(\frac{x}{\sin x} \right) \times \lim_{x \rightarrow 0} (a + \cos x) \\ &= \frac{1}{b} \times \frac{1}{\left(\lim_{x \rightarrow 0} \frac{\sin x}{x} \right)} \times \lim_{x \rightarrow 0} (a + \cos x) \\ &= \frac{1}{b} \times (a + \cos 0) \quad \left[\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1 \right] \\ &= \frac{a+1}{b} \end{aligned}$$

Question 19:

Evaluate the Given limit: $\lim_{x \rightarrow 0} x \sec x$

Answer

$$\lim_{x \rightarrow 0} x \sec x = \lim_{x \rightarrow 0} \frac{x}{\cos x} = \frac{0}{\cos 0} = \frac{0}{1} = 0$$

Question 20:

Evaluate the Given limit: $\lim_{x \rightarrow 0} \frac{\sin ax + bx}{ax + \sin bx}$ $a, b, a + b \neq 0$

Answer

$\frac{0}{0}$.

At $x = 0$, the value of the given function takes the form $\frac{0}{0}$.

Now,

$$\begin{aligned}
& \lim_{x \rightarrow 0} \frac{\sin ax + bx}{ax + \sin bx} \\
&= \lim_{x \rightarrow 0} \frac{\left(\frac{\sin ax}{ax}\right)ax + bx}{ax + bx\left(\frac{\sin bx}{bx}\right)} \\
&= \frac{\left(\lim_{ax \rightarrow 0} \frac{\sin ax}{ax}\right) \times \lim_{x \rightarrow 0} (ax) + \lim_{x \rightarrow 0} bx}{\lim_{x \rightarrow 0} ax + \lim_{x \rightarrow 0} bx \left(\lim_{bx \rightarrow 0} \frac{\sin bx}{bx}\right)} \quad [\text{As } x \rightarrow 0 \Rightarrow ax \rightarrow 0 \text{ and } bx \rightarrow 0] \\
&= \frac{\lim_{x \rightarrow 0} (ax) + \lim_{x \rightarrow 0} bx}{\lim_{x \rightarrow 0} ax + \lim_{x \rightarrow 0} bx} \quad \left[\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1 \right] \\
&= \frac{\lim_{x \rightarrow 0} (ax + bx)}{\lim_{x \rightarrow 0} (ax + bx)} \\
&= \lim_{x \rightarrow 0} (1) \\
&= 1
\end{aligned}$$

Question 21:

Evaluate the Given limit: $\lim_{x \rightarrow 0} (\operatorname{cosec} x - \cot x)$

Answer

At $x = 0$, the value of the given function takes the form $\infty - \infty$.

Now,

$$\lim_{x \rightarrow 0} (\operatorname{cosec} x - \cot x)$$

$$= \lim_{x \rightarrow 0} \left(\frac{1}{\sin x} - \frac{\cos x}{\sin x} \right)$$

$$= \lim_{x \rightarrow 0} \left(\frac{1 - \cos x}{\sin x} \right)$$

$$= \lim_{x \rightarrow 0} \frac{\left(\frac{1 - \cos x}{x} \right)}{\left(\frac{\sin x}{x} \right)}$$

$$= \frac{\lim_{x \rightarrow 0} \frac{1 - \cos x}{x}}{\lim_{x \rightarrow 0} \frac{\sin x}{x}}$$

$$= \frac{0}{1} \quad \left[\lim_{x \rightarrow 0} \frac{1 - \cos x}{x} = 0 \text{ and } \lim_{x \rightarrow 0} \frac{\sin x}{x} = 1 \right]$$

$$= 0$$

Question 22:

$$\lim_{x \rightarrow \frac{\pi}{2}} \frac{\tan 2x}{x - \frac{\pi}{2}}$$

Answer

$$\lim_{x \rightarrow \frac{\pi}{2}} \frac{\tan 2x}{x - \frac{\pi}{2}}$$

At $x = \frac{\pi}{2}$, the value of the given function takes the form $\frac{0}{0}$.

Now, put $x - \frac{\pi}{2} = y$ so that $x \rightarrow \frac{\pi}{2}, y \rightarrow 0$.

$$\begin{aligned}
\therefore \lim_{x \rightarrow \frac{\pi}{2}} \frac{\tan 2x}{x - \frac{\pi}{2}} &= \lim_{y \rightarrow 0} \frac{\tan 2\left(y + \frac{\pi}{2}\right)}{y} \\
&= \lim_{y \rightarrow 0} \frac{\tan(\pi + 2y)}{y} \\
&= \lim_{y \rightarrow 0} \frac{\tan 2y}{y} \quad [\tan(\pi + 2y) = \tan 2y] \\
&= \lim_{y \rightarrow 0} \frac{\sin 2y}{y \cos 2y} \\
&= \lim_{y \rightarrow 0} \left(\frac{\sin 2y}{2y} \times \frac{2}{\cos 2y} \right) \\
&= \left(\lim_{2y \rightarrow 0} \frac{\sin 2y}{2y} \right) \times \lim_{y \rightarrow 0} \left(\frac{2}{\cos 2y} \right) \quad [y \rightarrow 0 \Rightarrow 2y \rightarrow 0] \\
&= 1 \times \frac{2}{\cos 0} \quad \left[\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1 \right] \\
&= 1 \times \frac{2}{1} \\
&= 2
\end{aligned}$$

Question 23:

Find $\lim_{x \rightarrow 0} f(x)$ and $\lim_{x \rightarrow 1} f(x)$, where $f(x) = \begin{cases} 2x+3, & x \leq 0 \\ 3(x+1), & x > 0 \end{cases}$

Answer

The given function is

$$f(x) = \begin{cases} 2x+3, & x \leq 0 \\ 3(x+1), & x > 0 \end{cases}$$

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0} [2x+3] = 2(0)+3 = 3$$

$$\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0} 3(x+1) = 3(0+1) = 3$$

$$\therefore \lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0} f(x) = 3$$

$$\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1} 3(x+1) = 3(1+1) = 6$$

$$\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1} 3(x+1) = 3(1+1) = 6$$

$$\therefore \lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1} f(x) = 6$$

Question 24:

$$\text{Find } \lim_{x \rightarrow 1} f(x), \text{ where } f(x) = \begin{cases} x^2 - 1, & x \leq 1 \\ -x^2 - 1, & x > 1 \end{cases}$$

Answer

The given function is

$$f(x) = \begin{cases} x^2 - 1, & x \leq 1 \\ -x^2 - 1, & x > 1 \end{cases}$$

$$\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1} [x^2 - 1] = 1^2 - 1 = 1 - 1 = 0$$

$$\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1} [-x^2 - 1] = -1^2 - 1 = -1 - 1 = -2$$

It is observed that $\lim_{x \rightarrow 1^-} f(x) \neq \lim_{x \rightarrow 1^+} f(x)$.

Hence, $\lim_{x \rightarrow 1} f(x)$ does not exist.

Question 25:

$$\text{Evaluate } \lim_{x \rightarrow 0} f(x), \text{ where } f(x) = \begin{cases} \frac{|x|}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

Answer

The given function is

$$f(x) = \begin{cases} \frac{|x|}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

$$\begin{aligned}\lim_{x \rightarrow 0^-} f(x) &= \lim_{x \rightarrow 0^-} \left[\frac{|x|}{x} \right] \\ &= \lim_{x \rightarrow 0^-} \left(\frac{-x}{x} \right) \quad [\text{When } x \text{ is negative, } |x| = -x] \\ &= \lim_{x \rightarrow 0^-} (-1) \\ &= -1\end{aligned}$$

$$\begin{aligned}\lim_{x \rightarrow 0^+} f(x) &= \lim_{x \rightarrow 0^+} \left[\frac{|x|}{x} \right] \\ &= \lim_{x \rightarrow 0^+} \left[\frac{x}{x} \right] \quad [\text{When } x \text{ is positive, } |x| = x] \\ &= \lim_{x \rightarrow 0^+} (1) \\ &= 1\end{aligned}$$

It is observed that $\lim_{x \rightarrow 0^-} f(x) \neq \lim_{x \rightarrow 0^+} f(x)$.

Hence, $\lim_{x \rightarrow 0} f(x)$ does not exist.

Question 26:

$$\text{Find } \lim_{x \rightarrow 0} f(x), \text{ where } f(x) = \begin{cases} \frac{x}{|x|}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

Answer

The given function is

$$f(x) = \begin{cases} \frac{x}{|x|}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

$$\begin{aligned} \lim_{x \rightarrow 0^-} f(x) &= \lim_{x \rightarrow 0^-} \left[\frac{x}{|x|} \right] \\ &= \lim_{x \rightarrow 0} \left[\frac{x}{-x} \right] && \text{[When } x < 0, |x| = -x \text{]} \\ &= \lim_{x \rightarrow 0} (-1) \\ &= -1 \end{aligned}$$

$$\begin{aligned} \lim_{x \rightarrow 0^+} f(x) &= \lim_{x \rightarrow 0^+} \left[\frac{x}{|x|} \right] \\ &= \lim_{x \rightarrow 0} \left[\frac{x}{x} \right] && \text{[When } x > 0, |x| = x \text{]} \\ &= \lim_{x \rightarrow 0} (1) \\ &= 1 \end{aligned}$$

It is observed that $\lim_{x \rightarrow 0^-} f(x) \neq \lim_{x \rightarrow 0^+} f(x)$.

Hence, $\lim_{x \rightarrow 0} f(x)$ does not exist.

Question 27:

Find $\lim_{x \rightarrow 5} f(x)$, where $f(x) = |x| - 5$

Answer

The given function is $f(x) = |x| - 5$.

$$\begin{aligned}\lim_{x \rightarrow 5^-} f(x) &= \lim_{x \rightarrow 5^-} [|x| - 5] \\ &= \lim_{x \rightarrow 5} (x - 5) \quad [\text{When } x > 0, |x| = x] \\ &= 5 - 5 \\ &= 0\end{aligned}$$

$$\begin{aligned}\lim_{x \rightarrow 5^+} f(x) &= \lim_{x \rightarrow 5^+} (|x| - 5) \\ &= \lim_{x \rightarrow 5} (x - 5) \quad [\text{When } x > 0, |x| = x] \\ &= 5 - 5 \\ &= 0\end{aligned}$$

$$\therefore \lim_{x \rightarrow 5^-} f(x) = \lim_{x \rightarrow 5^+} f(x) = 0$$

$$\text{Hence, } \lim_{x \rightarrow 5} f(x) = 0$$

Question 28:

Suppose $f(x) = \begin{cases} a + bx, & x < 1 \\ 4, & x = 1 \\ b - ax & x > 1 \end{cases}$ and if $\lim_{x \rightarrow 1} f(x) = f(1)$ what are possible values of a and b ?

Answer

The given function is

$$f(x) = \begin{cases} a + bx, & x < 1 \\ 4, & x = 1 \\ b - ax & x > 1 \end{cases}$$

$$\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1} (a + bx) = a + b$$

$$\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1} (b - ax) = b - a$$

$$f(1) = 4$$

$$\text{It is given that } \lim_{x \rightarrow 1} f(x) = f(1).$$

$$\therefore \lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1} f(x) = f(1)$$

$$\Rightarrow a + b = 4 \text{ and } b - a = 4$$

On solving these two equations, we obtain $a = 0$ and $b = 4$.

Thus, the respective possible values of a and b are 0 and 4.

Question 29:

Let a_1, a_2, \dots, a_n be fixed real numbers and define a function

$$f(x) = (x - a_1)(x - a_2) \dots (x - a_n)$$

What is $\lim_{x \rightarrow a_1} f(x)$? For some $a \neq a_1, a_2, \dots, a_n$, compute $\lim_{x \rightarrow a} f(x)$.

Answer

The given function is $f(x) = (x - a_1)(x - a_2) \dots (x - a_n)$.

$$\begin{aligned} \lim_{x \rightarrow a_1} f(x) &= \lim_{x \rightarrow a_1} [(x - a_1)(x - a_2) \dots (x - a_n)] \\ &= \left[\lim_{x \rightarrow a_1} (x - a_1) \right] \left[\lim_{x \rightarrow a_1} (x - a_2) \right] \dots \left[\lim_{x \rightarrow a_1} (x - a_n) \right] \\ &= (a_1 - a_1)(a_1 - a_2) \dots (a_1 - a_n) = 0 \end{aligned}$$

$$\therefore \lim_{x \rightarrow a_1} f(x) = 0$$

$$\begin{aligned} \text{Now, } \lim_{x \rightarrow a} f(x) &= \lim_{x \rightarrow a} [(x - a_1)(x - a_2) \dots (x - a_n)] \\ &= \left[\lim_{x \rightarrow a} (x - a_1) \right] \left[\lim_{x \rightarrow a} (x - a_2) \right] \dots \left[\lim_{x \rightarrow a} (x - a_n) \right] \\ &= (a - a_1)(a - a_2) \dots (a - a_n) \end{aligned}$$

$$\therefore \lim_{x \rightarrow a} f(x) = (a - a_1)(a - a_2) \dots (a - a_n)$$

Question 30:

$$\text{If } f(x) = \begin{cases} |x| + 1, & x < 0 \\ 0, & x = 0 \\ |x| - 1, & x > 0 \end{cases}$$

For what value (s) of a does $\lim_{x \rightarrow a} f(x)$ exists?

Answer

The given function is

$$f(x) = \begin{cases} |x| + 1, & x < 0 \\ 0, & x = 0 \\ |x| - 1, & x > 0 \end{cases}$$

When $a = 0$,

$$\begin{aligned} \lim_{x \rightarrow 0^-} f(x) &= \lim_{x \rightarrow 0^-} (|x| + 1) \\ &= \lim_{x \rightarrow 0} (-x + 1) && [\text{If } x < 0, |x| = -x] \\ &= -0 + 1 \\ &= 1 \end{aligned}$$

$$\begin{aligned} \lim_{x \rightarrow 0^+} f(x) &= \lim_{x \rightarrow 0^+} (|x| - 1) \\ &= \lim_{x \rightarrow 0} (x - 1) && [\text{If } x > 0, |x| = x] \\ &= 0 - 1 \\ &= -1 \end{aligned}$$

Here, it is observed that $\lim_{x \rightarrow 0^-} f(x) \neq \lim_{x \rightarrow 0^+} f(x)$.

$\therefore \lim_{x \rightarrow 0} f(x)$ does not exist.

When $a < 0$,

$$\begin{aligned} \lim_{x \rightarrow a^-} f(x) &= \lim_{x \rightarrow a^-} (|x| + 1) \\ &= \lim_{x \rightarrow a} (-x + 1) && [x < a < 0 \Rightarrow |x| = -x] \\ &= -a + 1 \end{aligned}$$

$$\begin{aligned} \lim_{x \rightarrow a^+} f(x) &= \lim_{x \rightarrow a^+} (|x| + 1) \\ &= \lim_{x \rightarrow a} (-x + 1) && [a < x < 0 \Rightarrow |x| = -x] \\ &= -a + 1 \end{aligned}$$

$$\therefore \lim_{x \rightarrow a^-} f(x) = \lim_{x \rightarrow a^+} f(x) = -a + 1$$

Thus, limit of $f(x)$ exists at $x = a$, where $a < 0$.

When $a > 0$

$$\begin{aligned}\lim_{x \rightarrow a^-} f(x) &= \lim_{x \rightarrow a^-} (|x| - 1) \\ &= \lim_{x \rightarrow a} (x - 1) \quad [0 < x < a \Rightarrow |x| = x] \\ &= a - 1\end{aligned}$$

$$\begin{aligned}\lim_{x \rightarrow a^+} f(x) &= \lim_{x \rightarrow a^+} (|x| - 1) \\ &= \lim_{x \rightarrow a} (x - 1) \quad [0 < a < x \Rightarrow |x| = x] \\ &= a - 1\end{aligned}$$

$$\therefore \lim_{x \rightarrow a^-} f(x) = \lim_{x \rightarrow a^+} f(x) = a - 1$$

Thus, limit of $f(x)$ exists at $x = a$, where $a > 0$.

Thus, $\lim_{x \rightarrow a} f(x)$ exists for all $a \neq 0$.

Question 31:

If the function $f(x)$ satisfies $\lim_{x \rightarrow 1} \frac{f(x) - 2}{x^2 - 1} = \pi$, evaluate $\lim_{x \rightarrow 1} f(x)$.

Answer

$$\begin{aligned}\lim_{x \rightarrow 1} \frac{f(x) - 2}{x^2 - 1} &= \pi \\ \Rightarrow \frac{\lim_{x \rightarrow 1} (f(x) - 2)}{\lim_{x \rightarrow 1} (x^2 - 1)} &= \pi \\ \Rightarrow \lim_{x \rightarrow 1} (f(x) - 2) &= \pi \lim_{x \rightarrow 1} (x^2 - 1) \\ \Rightarrow \lim_{x \rightarrow 1} (f(x) - 2) &= \pi(1^2 - 1) \\ \Rightarrow \lim_{x \rightarrow 1} (f(x) - 2) &= 0 \\ \Rightarrow \lim_{x \rightarrow 1} f(x) - \lim_{x \rightarrow 1} 2 &= 0 \\ \Rightarrow \lim_{x \rightarrow 1} f(x) - 2 &= 0 \\ \therefore \lim_{x \rightarrow 1} f(x) &= 2\end{aligned}$$

Question 32:

$$f(x) = \begin{cases} mx^2 + n, & x < 0 \\ nx + m, & 0 \leq x \leq 1 \\ nx^3 + m, & x > 1 \end{cases}$$

If $\lim_{x \rightarrow 0} f(x)$ and $\lim_{x \rightarrow 1} f(x)$ exist? . For what integers m and n does $\lim_{x \rightarrow 0} f(x)$ and

$\lim_{x \rightarrow 1} f(x)$ exist?

Answer

The given function is

$$f(x) = \begin{cases} mx^2 + n, & x < 0 \\ nx + m, & 0 \leq x \leq 1 \\ nx^3 + m, & x > 1 \end{cases}$$

$$\begin{aligned} \lim_{x \rightarrow 0^-} f(x) &= \lim_{x \rightarrow 0^-} (mx^2 + n) \\ &= m(0)^2 + n \\ &= n \end{aligned}$$

$$\begin{aligned} \lim_{x \rightarrow 0^+} f(x) &= \lim_{x \rightarrow 0^+} (nx + m) \\ &= n(0) + m \\ &= m. \end{aligned}$$

Thus, $\lim_{x \rightarrow 0} f(x)$ exists if $m = n$.

$$\begin{aligned} \lim_{x \rightarrow 1^-} f(x) &= \lim_{x \rightarrow 1^-} (nx + m) \\ &= n(1) + m \\ &= m + n \end{aligned}$$

$$\begin{aligned} \lim_{x \rightarrow 1^+} f(x) &= \lim_{x \rightarrow 1^+} (nx^3 + m) \\ &= n(1)^3 + m \\ &= m + n \end{aligned}$$

$$\therefore \lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1} f(x).$$

Thus, $\lim_{x \rightarrow 1} f(x)$ exists for any integral value of m and n .

Exercise 13.2

Question 1:

Find the derivative of $x^2 - 2$ at $x = 10$.

Answer

Let $f(x) = x^2 - 2$. Accordingly,

$$\begin{aligned}f'(10) &= \lim_{h \rightarrow 0} \frac{f(10+h) - f(10)}{h} \\&= \lim_{h \rightarrow 0} \frac{[(10+h)^2 - 2] - (10^2 - 2)}{h} \\&= \lim_{h \rightarrow 0} \frac{10^2 + 2 \cdot 10 \cdot h + h^2 - 2 - 10^2 + 2}{h} \\&= \lim_{h \rightarrow 0} \frac{20h + h^2}{h} \\&= \lim_{h \rightarrow 0} (20 + h) = (20 + 0) = 20\end{aligned}$$

Thus, the derivative of $x^2 - 2$ at $x = 10$ is 20.

Question 2:

Find the derivative of $99x$ at $x = 100$.

Answer

Let $f(x) = 99x$. Accordingly,

$$\begin{aligned}f'(100) &= \lim_{h \rightarrow 0} \frac{f(100+h) - f(100)}{h} \\&= \lim_{h \rightarrow 0} \frac{99(100+h) - 99(100)}{h} \\&= \lim_{h \rightarrow 0} \frac{99 \times 100 + 99h - 99 \times 100}{h} \\&= \lim_{h \rightarrow 0} \frac{99h}{h} \\&= \lim_{h \rightarrow 0} (99) = 99\end{aligned}$$

Thus, the derivative of $99x$ at $x = 100$ is 99.

Question 3:

Find the derivative of x at $x = 1$.

Answer

Let $f(x) = x$. Accordingly,

$$\begin{aligned}f'(1) &= \lim_{h \rightarrow 0} \frac{f(1+h) - f(1)}{h} \\&= \lim_{h \rightarrow 0} \frac{(1+h) - 1}{h} \\&= \lim_{h \rightarrow 0} \frac{h}{h} \\&= \lim_{h \rightarrow 0} (1) \\&= 1\end{aligned}$$

Thus, the derivative of x at $x = 1$ is 1.

Question 4:

Find the derivative of the following functions from first principle.

(i) $x^3 - 27$ (ii) $(x - 1)(x - 2)$

(ii) $\frac{1}{x^2}$ (iv) $\frac{x+1}{x-1}$

Answer

(i) Let $f(x) = x^3 - 27$. Accordingly, from the first principle,

$$\begin{aligned}f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\&= \lim_{h \rightarrow 0} \frac{[(x+h)^3 - 27] - (x^3 - 27)}{h} \\&= \lim_{h \rightarrow 0} \frac{x^3 + h^3 + 3x^2h + 3xh^2 - x^3}{h} \\&= \lim_{h \rightarrow 0} \frac{h^3 + 3x^2h + 3xh^2}{h} \\&= \lim_{h \rightarrow 0} (h^2 + 3x^2 + 3xh) \\&= 0 + 3x^2 + 0 = 3x^2\end{aligned}$$

(ii) Let $f(x) = (x - 1)(x - 2)$. Accordingly, from the first principle,

$$\begin{aligned}
 f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{(x+h-1)(x+h-2) - (x-1)(x-2)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{(x^2 + hx - 2x + hx + h^2 - 2h - x - h + 2) - (x^2 - 2x - x + 2)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{(hx + hx + h^2 - 2h - h)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{2hx + h^2 - 3h}{h} \\
 &= \lim_{h \rightarrow 0} (2x + h - 3) \\
 &= (2x + 0 - 3) \\
 &= 2x - 3
 \end{aligned}$$

(iii) Let $f(x) = \frac{1}{x^2}$. Accordingly, from the first principle,

$$\begin{aligned}
 f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\frac{1}{(x+h)^2} - \frac{1}{x^2}}{h} \\
 &= \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{x^2 - (x+h)^2}{x^2(x+h)^2} \right] \\
 &= \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{x^2 - x^2 - h^2 - 2hx}{x^2(x+h)^2} \right] \\
 &= \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{-h^2 - 2hx}{x^2(x+h)^2} \right] \\
 &= \lim_{h \rightarrow 0} \left[\frac{-h - 2x}{x^2(x+h)^2} \right] \\
 &= \frac{0 - 2x}{x^2(x+0)^2} = \frac{-2}{x^3}
 \end{aligned}$$

(iv) Let $f(x) = \frac{x+1}{x-1}$. Accordingly, from the first principle,

$$\begin{aligned}f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\&= \lim_{h \rightarrow 0} \frac{\left(\frac{x+h+1}{x+h-1} - \frac{x+1}{x-1}\right)}{h} \\&= \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{(x-1)(x+h+1) - (x+1)(x+h-1)}{(x-1)(x+h-1)} \right] \\&= \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{(x^2 + hx + x - x - h - 1) - (x^2 + hx - x + x + h - 1)}{(x-1)(x+h-1)} \right] \\&= \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{-2h}{(x-1)(x+h-1)} \right] \\&= \lim_{h \rightarrow 0} \left[\frac{-2}{(x-1)(x+h-1)} \right] \\&= \frac{-2}{(x-1)(x-1)} = \frac{-2}{(x-1)^2}\end{aligned}$$

Question 5:

For the function

$$f(x) = \frac{x^{100}}{100} + \frac{x^{99}}{99} + \dots + \frac{x^2}{2} + x + 1$$

Prove that $f'(1) = 100f'(0)$

Answer

The given function is

$$f(x) = \frac{x^{100}}{100} + \frac{x^{99}}{99} + \dots + \frac{x^2}{2} + x + 1$$

$$\frac{d}{dx} f(x) = \frac{d}{dx} \left[\frac{x^{100}}{100} + \frac{x^{99}}{99} + \dots + \frac{x^2}{2} + x + 1 \right]$$

$$\frac{d}{dx} f(x) = \frac{d}{dx} \left(\frac{x^{100}}{100} \right) + \frac{d}{dx} \left(\frac{x^{99}}{99} \right) + \dots + \frac{d}{dx} \left(\frac{x^2}{2} \right) + \frac{d}{dx} (x) + \frac{d}{dx} (1)$$

On using theorem $\frac{d}{dx}(x^n) = nx^{n-1}$, we obtain

$$\begin{aligned} \frac{d}{dx} f(x) &= \frac{100x^{99}}{100} + \frac{99x^{98}}{99} + \dots + \frac{2x}{2} + 1 + 0 \\ &= x^{99} + x^{98} + \dots + x + 1 \end{aligned}$$

$$\therefore f'(x) = x^{99} + x^{98} + \dots + x + 1$$

At $x = 0$,

$$f'(0) = 1$$

At $x = 1$,

$$f'(1) = 1^{99} + 1^{98} + \dots + 1 + 1 = \underbrace{[1 + 1 + \dots + 1 + 1]}_{100 \text{ terms}} = 1 \times 100 = 100$$

Thus, $f'(1) = 100 \times f'(0)$

Question 6:

Find the derivative of $x^n + ax^{n-1} + a^2x^{n-2} + \dots + a^{n-1}x + a^n$ for some fixed real number a .

Answer

$$\text{Let } f(x) = x^n + ax^{n-1} + a^2x^{n-2} + \dots + a^{n-1}x + a^n$$

$$\begin{aligned} \therefore f'(x) &= \frac{d}{dx} (x^n + ax^{n-1} + a^2x^{n-2} + \dots + a^{n-1}x + a^n) \\ &= \frac{d}{dx} (x^n) + a \frac{d}{dx} (x^{n-1}) + a^2 \frac{d}{dx} (x^{n-2}) + \dots + a^{n-1} \frac{d}{dx} (x) + a^n \frac{d}{dx} (1) \end{aligned}$$

On using theorem $\frac{d}{dx} x^n = nx^{n-1}$, we obtain

$$\begin{aligned} f'(x) &= nx^{n-1} + a(n-1)x^{n-2} + a^2(n-2)x^{n-3} + \dots + a^{n-1} + a^n (0) \\ &= nx^{n-1} + a(n-1)x^{n-2} + a^2(n-2)x^{n-3} + \dots + a^{n-1} \end{aligned}$$

Question 7:

For some constants a and b , find the derivative of

(i) $(x - a)(x - b)$ (ii) $(ax^2 + b)^2$ (iii) $\frac{x - a}{x - b}$

Answer

(i) Let $f(x) = (x - a)(x - b)$

$$\Rightarrow f(x) = x^2 - (a + b)x + ab$$

$$\begin{aligned} \therefore f'(x) &= \frac{d}{dx}(x^2 - (a + b)x + ab) \\ &= \frac{d}{dx}(x^2) - (a + b)\frac{d}{dx}(x) + \frac{d}{dx}(ab) \end{aligned}$$

On using theorem $\frac{d}{dx}(x^n) = nx^{n-1}$, we obtain

$$f'(x) = 2x - (a + b) + 0 = 2x - a - b$$

(ii) Let $f(x) = (ax^2 + b)^2$

$$\Rightarrow f(x) = a^2x^4 + 2abx^2 + b^2$$

$$\therefore f'(x) = \frac{d}{dx}(a^2x^4 + 2abx^2 + b^2) = a^2\frac{d}{dx}(x^4) + 2ab\frac{d}{dx}(x^2) + \frac{d}{dx}(b^2)$$

On using theorem $\frac{d}{dx}x^n = nx^{n-1}$, we obtain

$$\begin{aligned} f'(x) &= a^2(4x^3) + 2ab(2x) + b^2(0) \\ &= 4a^2x^3 + 4abx \\ &= 4ax(ax^2 + b) \end{aligned}$$

(iii) Let $f(x) = \frac{(x - a)}{(x - b)}$

$$\Rightarrow f'(x) = \frac{d}{dx}\left(\frac{x - a}{x - b}\right)$$

By quotient rule,

$$\begin{aligned}
 f'(x) &= \frac{(x-b)\frac{d}{dx}(x-a) - (x-a)\frac{d}{dx}(x-b)}{(x-b)^2} \\
 &= \frac{(x-b)(1) - (x-a)(1)}{(x-b)^2} \\
 &= \frac{x-b-x+a}{(x-b)^2} \\
 &= \frac{a-b}{(x-b)^2}
 \end{aligned}$$

Question 8:

Find the derivative of $\frac{x^n - a^n}{x - a}$ for some constant a .

Answer

$$\begin{aligned}
 \text{Let } f(x) &= \frac{x^n - a^n}{x - a} \\
 \Rightarrow f'(x) &= \frac{d}{dx} \left(\frac{x^n - a^n}{x - a} \right)
 \end{aligned}$$

By quotient rule,

$$\begin{aligned}
 f'(x) &= \frac{(x-a)\frac{d}{dx}(x^n - a^n) - (x^n - a^n)\frac{d}{dx}(x-a)}{(x-a)^2} \\
 &= \frac{(x-a)(nx^{n-1} - 0) - (x^n - a^n)}{(x-a)^2} \\
 &= \frac{nx^n - anx^{n-1} - x^n + a^n}{(x-a)^2}
 \end{aligned}$$

Question 9:

Find the derivative of

(i) $2x - \frac{3}{4}$ (ii) $(5x^3 + 3x - 1)(x - 1)$

$$(iii) x^{-3} (5 + 3x) \quad (iv) x^5 (3 - 6x^{-9})$$

$$(v) x^{-4} (3 - 4x^{-5}) \quad (vi) \frac{2}{x+1} - \frac{x^2}{3x-1}$$

Answer

$$(i) \text{ Let } f(x) = 2x - \frac{3}{4}$$

$$\begin{aligned} f'(x) &= \frac{d}{dx} \left(2x - \frac{3}{4} \right) \\ &= 2 \frac{d}{dx} (x) - \frac{d}{dx} \left(\frac{3}{4} \right) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

$$(ii) \text{ Let } f(x) = (5x^3 + 3x - 1)(x - 1)$$

By Leibnitz product rule,

$$\begin{aligned} f'(x) &= (5x^3 + 3x - 1) \frac{d}{dx} (x - 1) + (x - 1) \frac{d}{dx} (5x^3 + 3x - 1) \\ &= (5x^3 + 3x - 1)(1) + (x - 1)(15x^2 + 3 - 0) \\ &= (5x^3 + 3x - 1) + (x - 1)(15x^2 + 3) \\ &= 5x^3 + 3x - 1 + 15x^3 + 3x - 15x^2 - 3 \\ &= 20x^3 - 15x^2 + 6x - 4 \end{aligned}$$

$$(iii) \text{ Let } f(x) = x^{-3} (5 + 3x)$$

By Leibnitz product rule,

$$\begin{aligned}
 f'(x) &= x^{-3} \frac{d}{dx}(5+3x) + (5+3x) \frac{d}{dx}(x^{-3}) \\
 &= x^{-3}(0+3) + (5+3x)(-3x^{-3-1}) \\
 &= x^{-3}(3) + (5+3x)(-3x^{-4}) \\
 &= 3x^{-3} - 15x^{-4} - 9x^{-3} \\
 &= -6x^{-3} - 15x^{-4} \\
 &= -3x^{-3} \left(2 + \frac{5}{x} \right) \\
 &= \frac{-3x^{-3}}{x} (2x+5) \\
 &= \frac{-3}{x^4} (5+2x)
 \end{aligned}$$

(iv) Let $f(x) = x^5 (3 - 6x^{-9})$

By Leibnitz product rule,

$$\begin{aligned}
 f'(x) &= x^5 \frac{d}{dx}(3-6x^{-9}) + (3-6x^{-9}) \frac{d}{dx}(x^5) \\
 &= x^5 \{0 - 6(-9)x^{-9-1}\} + (3-6x^{-9})(5x^4) \\
 &= x^5 (54x^{-10}) + 15x^4 - 30x^{-5} \\
 &= 54x^{-5} + 15x^4 - 30x^{-5} \\
 &= 24x^{-5} + 15x^4 \\
 &= 15x^4 + \frac{24}{x^5}
 \end{aligned}$$

(v) Let $f(x) = x^{-4} (3 - 4x^{-5})$

By Leibnitz product rule,

$$\begin{aligned}
 f'(x) &= x^{-4} \frac{d}{dx}(3-4x^{-5}) + (3-4x^{-5}) \frac{d}{dx}(x^{-4}) \\
 &= x^{-4} \{0 - 4(-5)x^{-5-1}\} + (3-4x^{-5})(-4)x^{-4-1} \\
 &= x^{-4} (20x^{-6}) + (3-4x^{-5})(-4x^{-5}) \\
 &= 20x^{-10} - 12x^{-5} + 16x^{-10} \\
 &= 36x^{-10} - 12x^{-5} \\
 &= -\frac{12}{x^5} + \frac{36}{x^{10}}
 \end{aligned}$$

$$(vi) \text{ Let } f(x) = \frac{2}{x+1} - \frac{x^2}{3x-1}$$

$$f'(x) = \frac{d}{dx} \left(\frac{2}{x+1} \right) - \frac{d}{dx} \left(\frac{x^2}{3x-1} \right)$$

By quotient rule,

$$\begin{aligned} f'(x) &= \left[\frac{(x+1) \frac{d}{dx}(2) - 2 \frac{d}{dx}(x+1)}{(x+1)^2} \right] - \left[\frac{(3x-1) \frac{d}{dx}(x^2) - x^2 \frac{d}{dx}(3x-1)}{(3x-1)^2} \right] \\ &= \left[\frac{(x+1)(0) - 2(1)}{(x+1)^2} \right] - \left[\frac{(3x-1)(2x) - (x^2)(3)}{(3x-1)^2} \right] \\ &= \frac{-2}{(x+1)^2} - \left[\frac{6x^2 - 2x - 3x^2}{(3x-1)^2} \right] \\ &= \frac{-2}{(x+1)^2} - \left[\frac{3x^2 - 2x^2}{(3x-1)^2} \right] \\ &= \frac{-2}{(x+1)^2} - \frac{x(3x-2)}{(3x-1)^2} \end{aligned}$$

Question 10:

Find the derivative of $\cos x$ from first principle.

Answer

Let $f(x) = \cos x$. Accordingly, from the first principle,

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{\cos(x+h) - \cos x}{h} \end{aligned}$$

$$\begin{aligned}
&= \lim_{h \rightarrow 0} \left[\frac{\cos x \cos h - \sin x \sin h - \cos x}{h} \right] \\
&= \lim_{h \rightarrow 0} \left[\frac{-\cos x(1 - \cos h) - \sin x \sin h}{h} \right] \\
&= \lim_{h \rightarrow 0} \left[\frac{-\cos x(1 - \cos h)}{h} - \frac{\sin x \sin h}{h} \right] \\
&= -\cos x \left(\lim_{h \rightarrow 0} \frac{1 - \cos h}{h} \right) - \sin x \lim_{h \rightarrow 0} \left(\frac{\sin h}{h} \right) \\
&= -\cos x(0) - \sin x(1) \quad \left[\lim_{h \rightarrow 0} \frac{1 - \cos h}{h} = 0 \text{ and } \lim_{h \rightarrow 0} \frac{\sin h}{h} = 1 \right] \\
&= -\sin x \\
\therefore f'(x) &= -\sin x
\end{aligned}$$

Question 11:

Find the derivative of the following functions:

- (i) $\sin x \cos x$ (ii) $\sec x$ (iii) $5 \sec x + 4 \cos x$
 (iv) $\operatorname{cosec} x$ (v) $3 \cot x + 5 \operatorname{cosec} x$
 (vi) $5 \sin x - 6 \cos x + 7$ (vii) $2 \tan x - 7 \sec x$

Answer

- (i) Let $f(x) = \sin x \cos x$. Accordingly, from the first principle,

$$\begin{aligned}
 f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\sin(x+h)\cos(x+h) - \sin x \cos x}{h} \\
 &= \lim_{h \rightarrow 0} \frac{1}{2h} [2\sin(x+h)\cos(x+h) - 2\sin x \cos x] \\
 &= \lim_{h \rightarrow 0} \frac{1}{2h} [\sin 2(x+h) - \sin 2x] \\
 &= \lim_{h \rightarrow 0} \frac{1}{2h} \left[2\cos \frac{2x+2h+2x}{2} \cdot \sin \frac{2x+2h-2x}{2} \right] \\
 &= \lim_{h \rightarrow 0} \frac{1}{h} \left[\cos \frac{4x+2h}{2} \sin \frac{2h}{2} \right] \\
 &= \lim_{h \rightarrow 0} \frac{1}{h} [\cos(2x+h)\sin h] \\
 &= \lim_{h \rightarrow 0} \cos(2x+h) \cdot \lim_{h \rightarrow 0} \frac{\sin h}{h} \\
 &= \cos(2x+0) \cdot 1 \\
 &= \cos 2x
 \end{aligned}$$

(ii) Let $f(x) = \sec x$. Accordingly, from the first principle,

$$\begin{aligned}
f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h \rightarrow 0} \frac{\sec(x+h) - \sec x}{h} \\
&= \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{1}{\cos(x+h)} - \frac{1}{\cos x} \right] \\
&= \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{\cos x - \cos(x+h)}{\cos x \cos(x+h)} \right] \\
&= \frac{1}{\cos x} \cdot \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{-2 \sin\left(\frac{x+x+h}{2}\right) \sin\left(\frac{x-x-h}{2}\right)}{\cos(x+h)} \right] \\
&= \frac{1}{\cos x} \cdot \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{-2 \sin\left(\frac{2x+h}{2}\right) \sin\left(-\frac{h}{2}\right)}{\cos(x+h)} \right] \\
&= \frac{1}{\cos x} \cdot \lim_{h \rightarrow 0} \left[\frac{\sin\left(\frac{2x+h}{2}\right) \frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}}{\cos(x+h)} \right] \\
&= \frac{1}{\cos x} \cdot \lim_{\frac{h}{2} \rightarrow 0} \frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)} \cdot \lim_{h \rightarrow 0} \frac{\sin\left(\frac{2x+h}{2}\right)}{\cos(x+h)} \\
&= \frac{1}{\cos x} \cdot 1 \cdot \frac{\sin x}{\cos x} \\
&= \sec x \tan x
\end{aligned}$$

(iii) Let $f(x) = 5 \sec x + 4 \cos x$. Accordingly, from the first principle,

$$\begin{aligned}
f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h \rightarrow 0} \frac{5 \sec(x+h) + 4 \cos(x+h) - [5 \sec x + 4 \cos x]}{h} \\
&= 5 \lim_{h \rightarrow 0} \frac{[\sec(x+h) - \sec x]}{h} + 4 \lim_{h \rightarrow 0} \frac{[\cos(x+h) - \cos x]}{h} \\
&= 5 \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{1}{\cos(x+h)} - \frac{1}{\cos x} \right] + 4 \lim_{h \rightarrow 0} \frac{1}{h} [\cos(x+h) - \cos x] \\
&= 5 \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{\cos x - \cos(x+h)}{\cos x \cos(x+h)} \right] + 4 \lim_{h \rightarrow 0} \frac{1}{h} [\cos x \cos h - \sin x \sin h - \cos x] \\
&= \frac{5}{\cos x} \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{-2 \sin\left(\frac{x+x+h}{2}\right) \sin\left(\frac{x-x-h}{2}\right)}{\cos(x+h)} \right] + 4 \lim_{h \rightarrow 0} \frac{1}{h} [-\cos x(1 - \cos h) - \sin x \sin h] \\
&= \frac{5}{\cos x} \cdot \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{-2 \sin\left(\frac{2x+h}{2}\right) \sin\left(-\frac{h}{2}\right)}{\cos(x+h)} \right] + 4 \left[-\cos x \lim_{h \rightarrow 0} \frac{(1 - \cos h)}{h} - \sin x \lim_{h \rightarrow 0} \frac{\sin h}{h} \right] \\
&= \frac{5}{\cos x} \cdot \lim_{h \rightarrow 0} \left[\frac{\sin\left(\frac{2x+h}{2}\right) \cdot \frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}}}{\cos(x+h)} \right] + 4 [(-\cos x) \cdot (0) - (\sin x) \cdot 1] \\
&= \frac{5}{\cos x} \cdot \left[\lim_{h \rightarrow 0} \frac{\sin\left(\frac{2x+h}{2}\right)}{\cos(x+h)} \cdot \lim_{h \rightarrow 0} \frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}} \right] - 4 \sin x \\
&= \frac{5}{\cos x} \cdot \frac{\sin x}{\cos x} \cdot 1 - 4 \sin x \\
&= 5 \sec x \tan x - 4 \sin x
\end{aligned}$$

(iv) Let $f(x) = \operatorname{cosec} x$. Accordingly, from the first principle,

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

$$f'(x) = \lim_{h \rightarrow 0} \frac{1}{h} [\operatorname{cosec}(x+h) - \operatorname{cosec} x]$$

$$= \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{1}{\sin(x+h)} - \frac{1}{\sin x} \right]$$

$$= \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{\sin x - \sin(x+h)}{\sin(x+h) \sin x} \right]$$

$$= \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{2 \cos \left(\frac{x+x+h}{2} \right) \cdot \sin \left(\frac{x-x-h}{2} \right)}{\sin(x+h) \sin x} \right]$$

$$= \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{2 \cos \left(\frac{2x+h}{2} \right) \sin \left(-\frac{h}{2} \right)}{\sin(x+h) \sin x} \right]$$

$$= \lim_{h \rightarrow 0} \frac{-\cos \left(\frac{2x+h}{2} \right) \cdot \frac{\sin \left(\frac{h}{2} \right)}{\left(\frac{h}{2} \right)}}{\sin(x+h) \sin x}$$

$$= \lim_{h \rightarrow 0} \left(\frac{-\cos \left(\frac{2x+h}{2} \right)}{\sin(x+h) \sin x} \right) \cdot \lim_{\frac{h}{2} \rightarrow 0} \frac{\sin \left(\frac{h}{2} \right)}{\left(\frac{h}{2} \right)}$$

$$= \left(\frac{-\cos x}{\sin x \sin x} \right) \cdot 1$$

$$= -\operatorname{cosec} x \cot x$$

(v) Let $f(x) = 3\cot x + 5\operatorname{cosec} x$. Accordingly, from the first principle,

$$\begin{aligned}
 f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{3 \cot(x+h) + 5 \operatorname{cosec}(x+h) - 3 \cot x - 5 \operatorname{cosec} x}{h} \\
 &= 3 \lim_{h \rightarrow 0} \frac{1}{h} [\cot(x+h) - \cot x] + 5 \lim_{h \rightarrow 0} \frac{1}{h} [\operatorname{cosec}(x+h) - \operatorname{cosec} x] \quad \dots(1)
 \end{aligned}$$

$$\begin{aligned}
 \text{Now, } \lim_{h \rightarrow 0} \frac{1}{h} [\cot(x+h) - \cot x] &= \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{\cos(x+h)}{\sin(x+h)} - \frac{\cos x}{\sin x} \right] \\
 &= \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{\cos(x+h) \sin x - \cos x \sin(x+h)}{\sin x \sin(x+h)} \right] \\
 &= \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{\sin(x-x-h)}{\sin x \sin(x+h)} \right] \\
 &= \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{\sin(-h)}{\sin x \sin(x+h)} \right] \\
 &= - \left(\lim_{h \rightarrow 0} \frac{\sin h}{h} \right) \cdot \left(\lim_{h \rightarrow 0} \frac{1}{\sin x \cdot \sin(x+h)} \right) \\
 &= -1 \cdot \frac{1}{\sin x \cdot \sin(x+0)} = \frac{-1}{\sin^2 x} = -\operatorname{cosec}^2 x \quad \dots(2)
 \end{aligned}$$

$$\begin{aligned}
& \lim_{h \rightarrow 0} \frac{1}{h} [\operatorname{cosec}(x+h) - \operatorname{cosec} x] \\
&= \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{1}{\sin(x+h)} - \frac{1}{\sin x} \right] \\
&= \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{\sin x - \sin(x+h)}{\sin(x+h) \sin x} \right] \\
&= \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{2 \cos\left(\frac{x+x+h}{2}\right) \cdot \sin\left(\frac{x-x-h}{2}\right)}{\sin(x+h) \sin x} \right] \\
&= \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{2 \cos\left(\frac{2x+h}{2}\right) \sin\left(-\frac{h}{2}\right)}{\sin(x+h) \sin x} \right] \\
&= \lim_{h \rightarrow 0} \frac{-\cos\left(\frac{2x+h}{2}\right) \cdot \frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}}{\sin(x+h) \sin x} \\
&= \lim_{h \rightarrow 0} \left(\frac{-\cos\left(\frac{2x+h}{2}\right)}{\sin(x+h) \sin x} \right) \cdot \lim_{\frac{h}{2} \rightarrow 0} \frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)} \\
&= \left(\frac{-\cos x}{\sin x \sin x} \right) \cdot 1 \\
&= -\operatorname{cosec} x \cot x \qquad \dots(3)
\end{aligned}$$

From (1), (2), and (3), we obtain

$$f'(x) = -3\operatorname{cosec}^2 x - 5\operatorname{cosec} x \cot x$$

(vi) Let $f(x) = 5\sin x - 6\cos x + 7$. Accordingly, from the first principle,

$$\begin{aligned}
f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h \rightarrow 0} \frac{1}{h} [5 \sin(x+h) - 6 \cos(x+h) + 7 - 5 \sin x + 6 \cos x - 7] \\
&= \lim_{h \rightarrow 0} \frac{1}{h} [5 \{\sin(x+h) - \sin x\} - 6 \{\cos(x+h) - \cos x\}] \\
&= 5 \lim_{h \rightarrow 0} \frac{1}{h} [\sin(x+h) - \sin x] - 6 \lim_{h \rightarrow 0} \frac{1}{h} [\cos(x+h) - \cos x] \\
&= 5 \lim_{h \rightarrow 0} \frac{1}{h} \left[2 \cos\left(\frac{x+h+x}{2}\right) \sin\left(\frac{x+h-x}{2}\right) \right] - 6 \lim_{h \rightarrow 0} \frac{\cos x \cos h - \sin x \sin h - \cos x}{h} \\
&= 5 \lim_{h \rightarrow 0} \frac{1}{h} \left[2 \cos\left(\frac{2x+h}{2}\right) \sin\left(\frac{h}{2}\right) \right] - 6 \lim_{h \rightarrow 0} \left[\frac{-\cos x(1 - \cos h) - \sin x \sin h}{h} \right] \\
&= 5 \lim_{h \rightarrow 0} \left[\cos\left(\frac{2x+h}{2}\right) \frac{\sin \frac{h}{2}}{\frac{h}{2}} \right] - 6 \lim_{h \rightarrow 0} \left[\frac{-\cos x(1 - \cos h)}{h} - \frac{\sin x \sin h}{h} \right] \\
&= 5 \left[\lim_{h \rightarrow 0} \cos\left(\frac{2x+h}{2}\right) \right] \left[\lim_{\frac{h}{2} \rightarrow 0} \frac{\sin \frac{h}{2}}{\frac{h}{2}} \right] - 6 \left[(-\cos x) \left(\lim_{h \rightarrow 0} \frac{1 - \cos h}{h} \right) - \sin x \lim_{h \rightarrow 0} \left(\frac{\sin h}{h} \right) \right] \\
&= 5 \cos x \cdot 1 - 6 [(-\cos x) \cdot (0) - \sin x \cdot 1] \\
&= 5 \cos x + 6 \sin x
\end{aligned}$$

(vii) Let $f(x) = 2 \tan x - 7 \sec x$. Accordingly, from the first principle,

$$\begin{aligned}
f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h \rightarrow 0} \frac{1}{h} [2 \tan(x+h) - 7 \sec(x+h) - 2 \tan x + 7 \sec x] \\
&= \lim_{h \rightarrow 0} \frac{1}{h} [2 \{ \tan(x+h) - \tan x \} - 7 \{ \sec(x+h) - \sec x \}] \\
&= 2 \lim_{h \rightarrow 0} \frac{1}{h} [\tan(x+h) - \tan x] - 7 \lim_{h \rightarrow 0} \frac{1}{h} [\sec(x+h) - \sec x] \\
&= 2 \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{\sin(x+h)}{\cos(x+h)} - \frac{\sin x}{\cos x} \right] - 7 \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{1}{\cos(x+h)} - \frac{1}{\cos x} \right] \\
&= 2 \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{\sin(x+h) \cos x - \sin x \cos(x+h)}{\cos x \cos(x+h)} \right] - 7 \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{\cos x - \cos(x+h)}{\cos x \cos(x+h)} \right] \\
&= 2 \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{\sin(x+h-x)}{\cos x \cos(x+h)} \right] - 7 \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{-2 \sin\left(\frac{x+x+h}{2}\right) \sin\left(\frac{x-x-h}{2}\right)}{\cos x \cos(x+h)} \right] \\
&= 2 \lim_{h \rightarrow 0} \left[\left(\frac{\sin h}{h} \right) \frac{1}{\cos x \cos(x+h)} \right] - 7 \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{-2 \sin\left(\frac{2x+h}{2}\right) \sin\left(-\frac{h}{2}\right)}{\cos x \cos(x+h)} \right] \\
&= 2 \left(\lim_{h \rightarrow 0} \frac{\sin h}{h} \right) \left(\lim_{h \rightarrow 0} \frac{1}{\cos x \cos(x+h)} \right) - 7 \left(\lim_{\frac{h}{2} \rightarrow 0} \frac{\sin \frac{h}{2}}{\frac{h}{2}} \right) \left(\lim_{h \rightarrow 0} \frac{\sin\left(\frac{2x+h}{2}\right)}{\cos x \cos(x+h)} \right) \\
&= 2 \cdot 1 \cdot \frac{1}{\cos x \cos x} - 7 \cdot 1 \cdot \left(\frac{\sin x}{\cos x \cos x} \right) \\
&= 2 \sec^2 x - 7 \sec x \tan x
\end{aligned}$$

Question 1:

Find the derivative of the following functions from first principle:

(i) $-x$ (ii) $(-x)^{-1}$ (iii) $\sin(x + 1)$

(iv) $\cos\left(x - \frac{\pi}{8}\right)$

Answer

(i) Let $f(x) = -x$. Accordingly, $f(x+h) = -(x+h)$

By first principle,

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{-(x+h) - (-x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{-x - h + x}{h} \\ &= \lim_{h \rightarrow 0} \frac{-h}{h} \\ &= \lim_{h \rightarrow 0} (-1) = -1 \end{aligned}$$

(ii) Let $f(x) = (-x)^{-1} = \frac{1}{-x} = \frac{-1}{x}$. Accordingly, $f(x+h) = \frac{-1}{(x+h)}$

By first principle,

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{-1}{x+h} - \left(\frac{-1}{x} \right) \right] \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{-1}{x+h} + \frac{1}{x} \right] \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{-x + (x+h)}{x(x+h)} \right] \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{-x + x + h}{x(x+h)} \right] \end{aligned}$$

$$\begin{aligned}
 &= \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{h}{x(x+h)} \right] \\
 &= \lim_{h \rightarrow 0} \frac{1}{x(x+h)} \\
 &= \frac{1}{x \cdot x} = \frac{1}{x^2}
 \end{aligned}$$

(iii) Let $f(x) = \sin(x+1)$. Accordingly, $f(x+h) = \sin(x+h+1)$

By first principle,

$$\begin{aligned}
 f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{1}{h} [\sin(x+h+1) - \sin(x+1)] \\
 &= \lim_{h \rightarrow 0} \frac{1}{h} \left[2 \cos\left(\frac{x+h+1+x+1}{2}\right) \sin\left(\frac{x+h+1-x-1}{2}\right) \right] \\
 &= \lim_{h \rightarrow 0} \frac{1}{h} \left[2 \cos\left(\frac{2x+h+2}{2}\right) \sin\left(\frac{h}{2}\right) \right] \\
 &= \lim_{h \rightarrow 0} \left[\cos\left(\frac{2x+h+2}{2}\right) \cdot \frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)} \right] \\
 &= \lim_{h \rightarrow 0} \cos\left(\frac{2x+h+2}{2}\right) \cdot \lim_{\frac{h}{2} \rightarrow 0} \frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)} \quad \left[\text{As } h \rightarrow 0 \Rightarrow \frac{h}{2} \rightarrow 0 \right] \\
 &= \cos\left(\frac{2x+0+2}{2}\right) \cdot 1 \quad \left[\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1 \right] \\
 &= \cos(x+1)
 \end{aligned}$$

(iv) Let $f(x) = \cos\left(x - \frac{\pi}{8}\right)$. Accordingly, $f(x+h) = \cos\left(x+h - \frac{\pi}{8}\right)$

By first principle,

$$\begin{aligned}
f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h \rightarrow 0} \frac{1}{h} \left[\cos\left(x+h-\frac{\pi}{8}\right) - \cos\left(x-\frac{\pi}{8}\right) \right] \\
&= \lim_{h \rightarrow 0} \frac{1}{h} \left[-2 \sin\left(\frac{x+h-\frac{\pi}{8} + x-\frac{\pi}{8}}{2}\right) \sin\left(\frac{x+h-\frac{\pi}{8} - x + \frac{\pi}{8}}{2}\right) \right] \\
&= \lim_{h \rightarrow 0} \frac{1}{h} \left[-2 \sin\left(\frac{2x+h-\frac{\pi}{4}}{2}\right) \sin\frac{h}{2} \right] \\
&= \lim_{h \rightarrow 0} \left[-\sin\left(\frac{2x+h-\frac{\pi}{4}}{2}\right) \frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)} \right] \\
&= \lim_{h \rightarrow 0} \left[-\sin\left(\frac{2x+h-\frac{\pi}{4}}{2}\right) \right] \cdot \lim_{\frac{h}{2} \rightarrow 0} \frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)} \quad \left[\text{As } h \rightarrow 0 \Rightarrow \frac{h}{2} \rightarrow 0 \right] \\
&= -\sin\left(\frac{2x+0-\frac{\pi}{4}}{2}\right) \cdot 1 \\
&= -\sin\left(x-\frac{\pi}{8}\right)
\end{aligned}$$

Question 2:

Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non-zero constants and m and n are integers): $(x + a)$

Answer

Let $f(x) = x + a$. Accordingly, $f(x+h) = x+h+a$

By first principle,

$$\begin{aligned}
 f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{x+h+a-x-a}{h} \\
 &= \lim_{h \rightarrow 0} \left(\frac{h}{h} \right) \\
 &= \lim_{h \rightarrow 0} (1) \\
 &= 1
 \end{aligned}$$

Question 3:

Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r

and s are fixed non-zero constants and m and n are integers): $(px+q)\left(\frac{r}{x}+s\right)$

Answer

$$\text{Let } f(x) = (px+q)\left(\frac{r}{x}+s\right)$$

By Leibnitz product rule,

$$\begin{aligned}
 f'(x) &= (px+q)\left(\frac{r}{x}+s\right)' + \left(\frac{r}{x}+s\right)(px+q)' \\
 &= (px+q)(rx^{-1}+s)' + \left(\frac{r}{x}+s\right)(p) \\
 &= (px+q)(-rx^{-2}) + \left(\frac{r}{x}+s\right)p \\
 &= (px+q)\left(\frac{-r}{x^2}\right) + \left(\frac{r}{x}+s\right)p \\
 &= \frac{-pr}{x} - \frac{qr}{x^2} + \frac{pr}{x} + ps \\
 &= ps - \frac{qr}{x^2}
 \end{aligned}$$

Question 4:

Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non-zero constants and m and n are integers): $(ax + b)(cx + d)^2$

Answer

$$\text{Let } f(x) = (ax + b)(cx + d)^2$$

By Leibnitz product rule,

$$\begin{aligned} f'(x) &= (ax + b) \frac{d}{dx} (cx + d)^2 + (cx + d)^2 \frac{d}{dx} (ax + b) \\ &= (ax + b) \frac{d}{dx} (c^2x^2 + 2cdx + d^2) + (cx + d)^2 \frac{d}{dx} (ax + b) \\ &= (ax + b) \left[\frac{d}{dx} (c^2x^2) + \frac{d}{dx} (2cdx) + \frac{d}{dx} d^2 \right] + (cx + d)^2 \left[\frac{d}{dx} ax + \frac{d}{dx} b \right] \\ &= (ax + b) (2c^2x + 2cd) + (cx + d)^2 a \\ &= 2c(ax + b)(cx + d) + a(cx + d)^2 \end{aligned}$$

Question 5:

Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r

and s are fixed non-zero constants and m and n are integers): $\frac{ax + b}{cx + d}$

Answer

$$\text{Let } f(x) = \frac{ax + b}{cx + d}$$

By quotient rule,

$$\begin{aligned}
 f'(x) &= \frac{(cx+d)\frac{d}{dx}(ax+b) - (ax+b)\frac{d}{dx}(cx+d)}{(cx+d)^2} \\
 &= \frac{(cx+d)(a) - (ax+b)(c)}{(cx+d)^2} \\
 &= \frac{acx + ad - acx - bc}{(cx+d)^2} \\
 &= \frac{ad - bc}{(cx+d)^2}
 \end{aligned}$$

Question 6:

Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non-zero constants and m and n are integers):

$$\frac{1 + \frac{1}{x}}{1 - \frac{1}{x}}$$

Answer

$$\text{Let } f(x) = \frac{1 + \frac{1}{x}}{1 - \frac{1}{x}} = \frac{\frac{x+1}{x}}{\frac{x-1}{x}} = \frac{x+1}{x-1}, \text{ where } x \neq 0$$

By quotient rule,

$$\begin{aligned}
 f'(x) &= \frac{(x-1)\frac{d}{dx}(x+1) - (x+1)\frac{d}{dx}(x-1)}{(x-1)^2}, \quad x \neq 0, 1 \\
 &= \frac{(x-1)(1) - (x+1)(1)}{(x-1)^2}, \quad x \neq 0, 1 \\
 &= \frac{x-1-x-1}{(x-1)^2}, \quad x \neq 0, 1 \\
 &= \frac{-2}{(x-1)^2}, \quad x \neq 0, 1
 \end{aligned}$$

Question 7:

Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r

and s are fixed non-zero constants and m and n are integers): $\frac{1}{ax^2 + bx + c}$

Answer

$$\text{Let } f(x) = \frac{1}{ax^2 + bx + c}$$

By quotient rule,

$$\begin{aligned} f'(x) &= \frac{(ax^2 + bx + c) \frac{d}{dx}(1) - \frac{d}{dx}(ax^2 + bx + c)}{(ax^2 + bx + c)^2} \\ &= \frac{(ax^2 + bx + c)(0) - (2ax + b)}{(ax^2 + bx + c)^2} \\ &= \frac{-(2ax + b)}{(ax^2 + bx + c)^2} \end{aligned}$$

Question 8:

Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r

and s are fixed non-zero constants and m and n are integers): $\frac{ax + b}{px^2 + qx + r}$

Answer

$$\text{Let } f(x) = \frac{ax + b}{px^2 + qx + r}$$

By quotient rule,

$$\begin{aligned}
 f'(x) &= \frac{(px^2 + qx + r) \frac{d}{dx}(ax + b) - (ax + b) \frac{d}{dx}(px^2 + qx + r)}{(px^2 + qx + r)^2} \\
 &= \frac{(px^2 + qx + r)(a) - (ax + b)(2px + q)}{(px^2 + qx + r)^2} \\
 &= \frac{apx^2 + aqx + ar - 2apx^2 - aqx - 2bpx - bq}{(px^2 + qx + r)^2} \\
 &= \frac{-apx^2 - 2bpx + ar - bq}{(px^2 + qx + r)^2}
 \end{aligned}$$

Question 9:

Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r

and s are fixed non-zero constants and m and n are integers): $\frac{px^2 + qx + r}{ax + b}$

Answer

$$\text{Let } f(x) = \frac{px^2 + qx + r}{ax + b}$$

By quotient rule,

$$\begin{aligned}
 f'(x) &= \frac{(ax + b) \frac{d}{dx}(px^2 + qx + r) - (px^2 + qx + r) \frac{d}{dx}(ax + b)}{(ax + b)^2} \\
 &= \frac{(ax + b)(2px + q) - (px^2 + qx + r)(a)}{(ax + b)^2} \\
 &= \frac{2apx^2 + aqx + 2bpx + bq - apx^2 - aqx - ar}{(ax + b)^2} \\
 &= \frac{apx^2 + 2bpx + bq - ar}{(ax + b)^2}
 \end{aligned}$$

Question 10:

Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r

and s are fixed non-zero constants and m and n are integers): $\frac{a}{x^4} - \frac{b}{x^2} + \cos x$

Answer

$$\text{Let } f(x) = \frac{a}{x^4} - \frac{b}{x^2} + \cos x$$

$$f'(x) = \frac{d}{dx} \left(\frac{a}{x^4} \right) - \frac{d}{dx} \left(\frac{b}{x^2} \right) + \frac{d}{dx} (\cos x)$$

$$= a \frac{d}{dx} (x^{-4}) - b \frac{d}{dx} (x^{-2}) + \frac{d}{dx} (\cos x)$$

$$= a(-4x^{-5}) - b(-2x^{-3}) + (-\sin x) \quad \left[\frac{d}{dx} (x^n) = nx^{n-1} \text{ and } \frac{d}{dx} (\cos x) = -\sin x \right]$$

$$= \frac{-4a}{x^5} + \frac{2b}{x^3} - \sin x$$

Question 11:

Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r

and s are fixed non-zero constants and m and n are integers): $4\sqrt{x} - 2$

Answer

$$\text{Let } f(x) = 4\sqrt{x} - 2$$

$$f'(x) = \frac{d}{dx} (4\sqrt{x} - 2) = \frac{d}{dx} (4\sqrt{x}) - \frac{d}{dx} (2)$$

$$= 4 \frac{d}{dx} \left(x^{\frac{1}{2}} \right) - 0 = 4 \left(\frac{1}{2} x^{\frac{1}{2}-1} \right)$$

$$= \left(2x^{-\frac{1}{2}} \right) = \frac{2}{\sqrt{x}}$$

Question 12:

Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r

and s are fixed non-zero constants and m and n are integers): $(ax + b)^n$

Answer

Let $f(x) = (ax + b)^n$. Accordingly, $f(x+h) = \{a(x+h) + b\}^n = (ax + ah + b)^n$

By first principle,

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{(ax + ah + b)^n - (ax + b)^n}{h} \\ &= \lim_{h \rightarrow 0} \frac{(ax + b)^n \left(1 + \frac{ah}{ax + b}\right)^n - (ax + b)^n}{h} \\ &= (ax + b)^n \lim_{h \rightarrow 0} \frac{\left(1 + \frac{ah}{ax + b}\right)^n - 1}{h} \\ &= (ax + b)^n \lim_{h \rightarrow 0} \frac{1}{n} \left[\left\{ 1 + n \left(\frac{ah}{ax + b}\right) + \frac{n(n-1)}{2} \left(\frac{ah}{ax + b}\right)^2 + \dots \right\} - 1 \right] \\ &\quad \text{(Using binomial theorem)} \\ &= (ax + b)^n \lim_{h \rightarrow 0} \frac{1}{h} \left[n \left(\frac{ah}{ax + b}\right) + \frac{n(n-1)a^2h^2}{2(ax + b)^2} + \dots \text{(Terms containing higher degrees of } h) \right] \\ &= (ax + b)^n \lim_{h \rightarrow 0} \left[\frac{na}{(ax + b)} + \frac{n(n-1)a^2h}{2(ax + b)^2} + \dots \right] \\ &= (ax + b)^n \left[\frac{na}{(ax + b)} + 0 \right] \\ &= na \frac{(ax + b)^n}{(ax + b)} \\ &= na(ax + b)^{n-1} \end{aligned}$$

Question 13:

Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non-zero constants and m and n are integers): $(ax + b)^n (cx + d)^m$

Answer

Let $f(x) = (ax + b)^n (cx + d)^m$

By Leibnitz product rule,

$$f'(x) = (ax+b)^n \frac{d}{dx}(cx+d)^m + (cx+d)^m \frac{d}{dx}(ax+b)^n \quad \dots(1)$$

$$\text{Now, let } f_1(x) = (cx+d)^m$$

$$f_1(x+h) = (cx+ch+d)^m$$

$$f_1'(x) = \lim_{h \rightarrow 0} \frac{f_1(x+h) - f_1(x)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{(cx+ch+d)^m - (cx+d)^m}{h}$$

$$= (cx+d)^m \lim_{h \rightarrow 0} \frac{1}{h} \left[\left(1 + \frac{ch}{cx+d} \right)^m - 1 \right]$$

$$= (cx+d)^m \lim_{h \rightarrow 0} \frac{1}{h} \left[\left(1 + \frac{mch}{(cx+d)} + \frac{m(m-1)}{2} \frac{(c^2h^2)}{(cx+d)^2} + \dots \right) - 1 \right]$$

$$= (cx+d)^m \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{mch}{(cx+d)} + \frac{m(m-1)c^2h^2}{2(cx+d)^2} + \dots (\text{Terms containing higher degrees of } h) \right]$$

$$= (cx+d)^m \lim_{h \rightarrow 0} \left[\frac{mc}{(cx+d)} + \frac{m(m-1)c^2h}{2(cx+d)^2} + \dots \right]$$

$$= (cx+d)^m \left[\frac{mc}{cx+d} + 0 \right]$$

$$= \frac{mc(cx+d)^m}{(cx+d)}$$

$$= mc(cx+d)^{m-1}$$

$$\frac{d}{dx}(cx+d)^m = mc(cx+d)^{m-1} \quad \dots(2)$$

$$\text{Similarly, } \frac{d}{dx}(ax+b)^n = na(ax+b)^{n-1} \quad \dots(3)$$

Therefore, from (1), (2), and (3), we obtain

$$\begin{aligned} f'(x) &= (ax+b)^n \left\{ mc(cx+d)^{m-1} \right\} + (cx+d)^m \left\{ na(ax+b)^{n-1} \right\} \\ &= (ax+b)^{n-1} (cx+d)^{m-1} [mc(ax+b) + na(cx+d)] \end{aligned}$$

Question 14:

Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non-zero constants and m and n are integers): $\sin(x + a)$

Answer

$$\text{Let } f(x) = \sin(x + a)$$

$$f(x + h) = \sin(x + h + a)$$

By first principle,

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{\sin(x+h+a) - \sin(x+a)}{h} \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \left[2 \cos\left(\frac{x+h+a+x+a}{2}\right) \sin\left(\frac{x+h+a-x-a}{2}\right) \right] \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \left[2 \cos\left(\frac{2x+2a+h}{2}\right) \sin\left(\frac{h}{2}\right) \right] \\ &= \lim_{h \rightarrow 0} \left[\cos\left(\frac{2x+2a+h}{2}\right) \left\{ \frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)} \right\} \right] \\ &= \lim_{h \rightarrow 0} \cos\left(\frac{2x+2a+h}{2}\right) \lim_{\frac{h}{2} \rightarrow 0} \left\{ \frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)} \right\} \quad \left[\text{As } h \rightarrow 0 \Rightarrow \frac{h}{2} \rightarrow 0 \right] \\ &= \cos\left(\frac{2x+2a}{2}\right) \times 1 \quad \left[\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1 \right] \\ &= \cos(x + a) \end{aligned}$$

Question 15:

Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non-zero constants and m and n are integers): $\operatorname{cosec} x \cot x$

Answer

$$\text{Let } f(x) = \operatorname{cosec} x \cot x$$

By Leibnitz product rule,

$$f'(x) = \operatorname{cosec} x (\cot x)' + \cot x (\operatorname{cosec} x)' \quad \dots(1)$$

Let $f_1(x) = \cot x$. Accordingly, $f_1(x+h) = \cot(x+h)$

By first principle,

$$\begin{aligned} f_1'(x) &= \lim_{h \rightarrow 0} \frac{f_1(x+h) - f_1(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{\cot(x+h) - \cot x}{h} \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \left(\frac{\cos(x+h)}{\sin(x+h)} - \frac{\cos x}{\sin x} \right) \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{\sin x \cos(x+h) - \cos x \sin(x+h)}{\sin x \sin(x+h)} \right] \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{\sin(x-x-h)}{\sin x \sin(x+h)} \right] \\ &= \frac{1}{\sin x} \cdot \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{\sin(-h)}{\sin(x+h)} \right] \\ &= \frac{-1}{\sin x} \cdot \left(\lim_{h \rightarrow 0} \frac{\sin h}{h} \right) \left(\lim_{h \rightarrow 0} \frac{1}{\sin(x+h)} \right) \\ &= \frac{-1}{\sin x} \cdot 1 \cdot \left(\frac{1}{\sin(x+0)} \right) \\ &= \frac{-1}{\sin^2 x} \\ &= -\operatorname{cosec}^2 x \end{aligned}$$

$$\therefore (\cot x)' = -\operatorname{cosec}^2 x \quad \dots(2)$$

Now, let $f_2(x) = \operatorname{cosec} x$. Accordingly, $f_2(x+h) = \operatorname{cosec}(x+h)$

By first principle,

$$\begin{aligned} f_2'(x) &= \lim_{h \rightarrow 0} \frac{f_2(x+h) - f_2(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{1}{h} [\operatorname{cosec}(x+h) - \operatorname{cosec} x] \end{aligned}$$

$$\begin{aligned}
 &= \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{1}{\sin(x+h)} - \frac{1}{\sin x} \right] \\
 &= \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{\sin x - \sin(x+h)}{\sin x \sin(x+h)} \right] \\
 &= \frac{1}{\sin x} \cdot \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{2 \cos\left(\frac{x+x+h}{2}\right) \sin\left(\frac{x-x-h}{2}\right)}{\sin(x+h)} \right] \\
 &= \frac{1}{\sin x} \cdot \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{2 \cos\left(\frac{2x+h}{2}\right) \sin\left(\frac{-h}{2}\right)}{\sin(x+h)} \right] \\
 &= \frac{1}{\sin x} \cdot \lim_{h \rightarrow 0} \left[\frac{-\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)} \cdot \frac{\cos\left(\frac{2x+h}{2}\right)}{\sin(x+h)} \right] \\
 &= \frac{-1}{\sin x} \cdot \lim_{h \rightarrow 0} \frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)} \cdot \lim_{h \rightarrow 0} \frac{\cos\left(\frac{2x+h}{2}\right)}{\sin(x+h)} \\
 &= \frac{-1}{\sin x} \cdot 1 \cdot \frac{\cos\left(\frac{2x+0}{2}\right)}{\sin(x+0)} \\
 &= \frac{-1}{\sin x} \cdot \frac{\cos x}{\sin x} \\
 &= -\operatorname{cosec} x \cdot \cot x
 \end{aligned}$$

$$\therefore (\operatorname{cosec} x)' = -\operatorname{cosec} x \cot x \quad \dots(3)$$

From (1), (2), and (3), we obtain

$$\begin{aligned}
 f'(x) &= \operatorname{cosec} x (-\operatorname{cosec}^2 x) + \cot x (-\operatorname{cosec} x \cot x) \\
 &= -\operatorname{cosec}^3 x - \cot^2 x \operatorname{cosec} x
 \end{aligned}$$

Question 16:

Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r

and s are fixed non-zero constants and m and n are integers): $\frac{\cos x}{1 + \sin x}$

Answer

$$\text{Let } f(x) = \frac{\cos x}{1 + \sin x}$$

By quotient rule,

$$\begin{aligned} f'(x) &= \frac{(1 + \sin x) \frac{d}{dx}(\cos x) - (\cos x) \frac{d}{dx}(1 + \sin x)}{(1 + \sin x)^2} \\ &= \frac{(1 + \sin x)(-\sin x) - (\cos x)(\cos x)}{(1 + \sin x)^2} \\ &= \frac{-\sin x - \sin^2 x - \cos^2 x}{(1 + \sin x)^2} \\ &= \frac{-\sin x - (\sin^2 x + \cos^2 x)}{(1 + \sin x)^2} \\ &= \frac{-\sin x - 1}{(1 + \sin x)^2} \\ &= \frac{-(1 + \sin x)}{(1 + \sin x)^2} \\ &= \frac{-1}{(1 + \sin x)} \end{aligned}$$

Question 17:

Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r

and s are fixed non-zero constants and m and n are integers): $\frac{\sin x + \cos x}{\sin x - \cos x}$

Answer

$$\text{Let } f(x) = \frac{\sin x + \cos x}{\sin x - \cos x}$$

By quotient rule,

$$\begin{aligned}
 f'(x) &= \frac{(\sin x - \cos x) \frac{d}{dx}(\sin x + \cos x) - (\sin x + \cos x) \frac{d}{dx}(\sin x - \cos x)}{(\sin x - \cos x)^2} \\
 &= \frac{(\sin x - \cos x)(\cos x - \sin x) - (\sin x + \cos x)(\cos x + \sin x)}{(\sin x - \cos x)^2} \\
 &= \frac{-(\sin x - \cos x)^2 - (\sin x + \cos x)^2}{(\sin x - \cos x)^2} \\
 &= \frac{-[\sin^2 x + \cos^2 x - 2 \sin x \cos x + \sin^2 x + \cos^2 x + 2 \sin x \cos x]}{(\sin x - \cos x)^2} \\
 &= \frac{-[1+1]}{(\sin x - \cos x)^2} \\
 &= \frac{-2}{(\sin x - \cos x)^2}
 \end{aligned}$$

Question 18:

Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r

and s are fixed non-zero constants and m and n are integers): $\frac{\sec x - 1}{\sec x + 1}$

Answer

Let $f(x) = \frac{\sec x - 1}{\sec x + 1}$

$$f(x) = \frac{\frac{1}{\cos x} - 1}{\frac{1}{\cos x} + 1} = \frac{1 - \cos x}{1 + \cos x}$$

By quotient rule,

$$\begin{aligned}
 f'(x) &= \frac{(1 + \cos x) \frac{d}{dx}(1 - \cos x) - (1 - \cos x) \frac{d}{dx}(1 + \cos x)}{(1 + \cos x)^2} \\
 &= \frac{(1 + \cos x)(\sin x) - (1 - \cos x)(-\sin x)}{(1 + \cos x)^2} \\
 &= \frac{\sin x + \cos x \sin x + \sin x - \sin x \cos x}{(1 + \cos x)^2} \\
 &= \frac{2 \sin x}{(1 + \cos x)^2} \\
 &= \frac{2 \sin x}{\left(1 + \frac{1}{\sec x}\right)^2} = \frac{2 \sin x}{\frac{(\sec x + 1)^2}{\sec^2 x}} \\
 &= \frac{2 \sin x \sec^2 x}{(\sec x + 1)^2} \\
 &= \frac{2 \sin x}{\cos x} \sec x \\
 &= \frac{2 \sec x \tan x}{(\sec x + 1)^2}
 \end{aligned}$$

Question 19:

Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non-zero constants and m and n are integers): $\sin^n x$

Answer

Let $y = \sin^n x$.

Accordingly, for $n = 1$, $y = \sin x$.

$$\therefore \frac{dy}{dx} = \cos x, \text{ i.e., } \frac{d}{dx} \sin x = \cos x$$

For $n = 2$, $y = \sin^2 x$.

$$\begin{aligned} \therefore \frac{dy}{dx} &= \frac{d}{dx}(\sin x \sin x) \\ &= (\sin x)' \sin x + \sin x (\sin x)' && \text{[By Leibnitz product rule]} \\ &= \cos x \sin x + \sin x \cos x \\ &= 2 \sin x \cos x && \dots(1) \end{aligned}$$

For $n = 3$, $y = \sin^3 x$.

$$\begin{aligned} \therefore \frac{dy}{dx} &= \frac{d}{dx}(\sin x \sin^2 x) \\ &= (\sin x)' \sin^2 x + \sin x (\sin^2 x)' && \text{[By Leibnitz product rule]} \\ &= \cos x \sin^2 x + \sin x (2 \sin x \cos x) && \text{[Using (1)]} \\ &= \cos x \sin^2 x + 2 \sin^2 x \cos x \\ &= 3 \sin^2 x \cos x \end{aligned}$$

We assert that $\frac{d}{dx}(\sin^n x) = n \sin^{(n-1)} x \cos x$

Let our assertion be true for $n = k$.

$$\text{i.e., } \frac{d}{dx}(\sin^k x) = k \sin^{(k-1)} x \cos x \quad \dots(2)$$

Consider

$$\begin{aligned} \frac{d}{dx}(\sin^{k+1} x) &= \frac{d}{dx}(\sin x \sin^k x) \\ &= (\sin x)' \sin^k x + \sin x (\sin^k x)' && \text{[By Leibnitz product rule]} \\ &= \cos x \sin^k x + \sin x (k \sin^{(k-1)} x \cos x) && \text{[Using (2)]} \\ &= \cos x \sin^k x + k \sin^k x \cos x \\ &= (k+1) \sin^k x \cos x \end{aligned}$$

Thus, our assertion is true for $n = k + 1$.

Hence, by mathematical induction, $\frac{d}{dx}(\sin^n x) = n \sin^{(n-1)} x \cos x$

Question 20:

Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r

$$\frac{a + b \sin x}{c + d \cos x}$$

and s are fixed non-zero constants and m and n are integers):

Answer

$$\text{Let } f(x) = \frac{a + b \sin x}{c + d \cos x}$$

By quotient rule,

$$\begin{aligned} f'(x) &= \frac{(c + d \cos x) \frac{d}{dx}(a + b \sin x) - (a + b \sin x) \frac{d}{dx}(c + d \cos x)}{(c + d \cos x)^2} \\ &= \frac{(c + d \cos x)(b \cos x) - (a + b \sin x)(-d \sin x)}{(c + d \cos x)^2} \\ &= \frac{cb \cos x + bd \cos^2 x + ad \sin x + bd \sin^2 x}{(c + d \cos x)^2} \\ &= \frac{bc \cos x + ad \sin x + bd(\cos^2 x + \sin^2 x)}{(c + d \cos x)^2} \\ &= \frac{bc \cos x + ad \sin x + bd}{(c + d \cos x)^2} \end{aligned}$$

Question 21:

Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r

$$\frac{\sin(x + a)}{\cos x}$$

and s are fixed non-zero constants and m and n are integers):

Answer

$$\text{Let } f(x) = \frac{\sin(x + a)}{\cos x}$$

By quotient rule,

$$f'(x) = \frac{\cos x \frac{d}{dx} [\sin(x+a)] - \sin(x+a) \frac{d}{dx} \cos x}{\cos^2 x}$$

$$f'(x) = \frac{\cos x \frac{d}{dx} [\sin(x+a)] - \sin(x+a)(-\sin x)}{\cos^2 x} \quad \dots (i)$$

Let $g(x) = \sin(x+a)$. Accordingly, $g(x+h) = \sin(x+h+a)$

By first principle,

$$\begin{aligned} g'(x) &= \lim_{h \rightarrow 0} \frac{g(x+h) - g(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{1}{h} [\sin(x+h+a) - \sin(x+a)] \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \left[2 \cos \left(\frac{x+h+a+x+a}{2} \right) \sin \left(\frac{x+h+a-x-a}{2} \right) \right] \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \left[2 \cos \left(\frac{2x+2a+h}{2} \right) \sin \left(\frac{h}{2} \right) \right] \\ &= \lim_{h \rightarrow 0} \left[\cos \left(\frac{2x+2a+h}{2} \right) \left\{ \frac{\sin \left(\frac{h}{2} \right)}{\left(\frac{h}{2} \right)} \right\} \right] \\ &= \lim_{h \rightarrow 0} \cos \left(\frac{2x+2a+h}{2} \right) \cdot \lim_{\frac{h}{2} \rightarrow 0} \left\{ \frac{\sin \left(\frac{h}{2} \right)}{\left(\frac{h}{2} \right)} \right\} \quad \left[\text{As } h \rightarrow 0 \Rightarrow \frac{h}{2} \rightarrow 0 \right] \\ &= \left(\cos \frac{2x+2a}{2} \right) \times 1 \quad \left[\lim_{h \rightarrow 0} \frac{\sin h}{h} = 1 \right] \\ &= \cos(x+a) \quad \dots (ii) \end{aligned}$$

From (i) and (ii), we obtain

$$\begin{aligned} f'(x) &= \frac{\cos x \cdot \cos(x+a) + \sin x \sin(x+a)}{\cos^2 x} \\ &= \frac{\cos(x+a-x)}{\cos^2 x} \\ &= \frac{\cos a}{\cos^2 x} \end{aligned}$$

Question 22:

Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non-zero constants and m and n are integers): $x^4 (5 \sin x - 3 \cos x)$

Answer

Let $f(x) = x^4 (5 \sin x - 3 \cos x)$

By product rule,

$$\begin{aligned} f'(x) &= x^4 \frac{d}{dx}(5 \sin x - 3 \cos x) + (5 \sin x - 3 \cos x) \frac{d}{dx}(x^4) \\ &= x^4 \left[5 \frac{d}{dx}(\sin x) - 3 \frac{d}{dx}(\cos x) \right] + (5 \sin x - 3 \cos x) \frac{d}{dx}(x^4) \\ &= x^4 [5 \cos x - 3(-\sin x)] + (5 \sin x - 3 \cos x)(4x^3) \\ &= x^3 [5x \cos x + 3x \sin x + 20 \sin x - 12 \cos x] \end{aligned}$$

Question 23:

Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non-zero constants and m and n are integers): $(x^2 + 1) \cos x$

Answer

Let $f(x) = (x^2 + 1) \cos x$

By product rule,

$$\begin{aligned} f'(x) &= (x^2 + 1) \frac{d}{dx}(\cos x) + \cos x \frac{d}{dx}(x^2 + 1) \\ &= (x^2 + 1)(-\sin x) + \cos x(2x) \\ &= -x^2 \sin x - \sin x + 2x \cos x \end{aligned}$$

Question 24:

Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non-zero constants and m and n are integers): $(ax^2 + \sin x)(p + q \cos x)$

Answer

Let $f(x) = (ax^2 + \sin x)(p + q \cos x)$

By product rule,

$$\begin{aligned}f'(x) &= (ax^2 + \sin x) \frac{d}{dx}(p + q \cos x) + (p + q \cos x) \frac{d}{dx}(ax^2 + \sin x) \\&= (ax^2 + \sin x)(-q \sin x) + (p + q \cos x)(2ax + \cos x) \\&= -q \sin x(ax^2 + \sin x) + (p + q \cos x)(2ax + \cos x)\end{aligned}$$

Question 25:

Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r

and s are fixed non-zero constants and m and n are integers): $(x + \cos x)(x - \tan x)$

Answer

Let $f(x) = (x + \cos x)(x - \tan x)$

By product rule,

$$\begin{aligned}f'(x) &= (x + \cos x) \frac{d}{dx}(x - \tan x) + (x - \tan x) \frac{d}{dx}(x + \cos x) \\&= (x + \cos x) \left[\frac{d}{dx}(x) - \frac{d}{dx}(\tan x) \right] + (x - \tan x)(1 - \sin x) \\&= (x + \cos x) \left[1 - \frac{d}{dx} \tan x \right] + (x - \tan x)(1 - \sin x) \quad \dots \text{(i)}\end{aligned}$$

Let $g(x) = \tan x$. Accordingly, $g(x+h) = \tan(x+h)$

By first principle,

$$\begin{aligned}
g'(x) &= \lim_{h \rightarrow 0} \frac{g(x+h) - g(x)}{h} \\
&= \lim_{h \rightarrow 0} \left(\frac{\tan(x+h) - \tan x}{h} \right) \\
&= \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{\sin(x+h)}{\cos(x+h)} - \frac{\sin x}{\cos x} \right] \\
&= \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{\sin(x+h)\cos x - \sin x\cos(x+h)}{\cos(x+h)\cos x} \right] \\
&= \frac{1}{\cos x} \cdot \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{\sin(x+h-x)}{\cos(x+h)} \right] \\
&= \frac{1}{\cos x} \cdot \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{\sin h}{\cos(x+h)} \right] \\
&= \frac{1}{\cos x} \cdot \left(\lim_{h \rightarrow 0} \frac{\sin h}{h} \right) \cdot \left(\lim_{h \rightarrow 0} \frac{1}{\cos(x+h)} \right) \\
&= \frac{1}{\cos x} \cdot 1 \cdot \frac{1}{\cos(x+0)} \\
&= \frac{1}{\cos^2 x} \\
&= \sec^2 x \quad \dots \text{(ii)}
\end{aligned}$$

Therefore, from (i) and (ii), we obtain

$$\begin{aligned}
f'(x) &= (x + \cos x)(1 - \sec^2 x) + (x - \tan x)(1 - \sin x) \\
&= (x + \cos x)(-\tan^2 x) + (x - \tan x)(1 - \sin x) \\
&= -\tan^2 x(x + \cos x) + (x - \tan x)(1 - \sin x)
\end{aligned}$$

Question 26:

Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r

and s are fixed non-zero constants and m and n are integers): $\frac{4x + 5 \sin x}{3x + 7 \cos x}$

Answer

Let $f(x) = \frac{4x + 5 \sin x}{3x + 7 \cos x}$

By quotient rule,

$$\begin{aligned}
 f'(x) &= \frac{(3x + 7 \cos x) \frac{d}{dx}(4x + 5 \sin x) - (4x + 5 \sin x) \frac{d}{dx}(3x + 7 \cos x)}{(3x + 7 \cos x)^2} \\
 &= \frac{(3x + 7 \cos x) \left[4 \frac{d}{dx}(x) + 5 \frac{d}{dx}(\sin x) \right] - (4x + 5 \sin x) \left[3 \frac{d}{dx}x + 7 \frac{d}{dx} \cos x \right]}{(3x + 7 \cos x)^2} \\
 &= \frac{(3x + 7 \cos x)(4 + 5 \cos x) - (4x + 5 \sin x)(3 - 7 \sin x)}{(3x + 7 \cos x)^2} \\
 &= \frac{12x + 15x \cos x + 28 \cos x + 35 \cos^2 x - 12x + 28x \sin x - 15 \sin x + 35 \sin^2 x}{(3x + 7 \cos x)^2} \\
 &= \frac{15x \cos x + 28 \cos x + 28x \sin x - 15 \sin x + 35(\cos^2 x + \sin^2 x)}{(3x + 7 \cos x)^2} \\
 &= \frac{35 + 15x \cos x + 28 \cos x + 28x \sin x - 15 \sin x}{(3x + 7 \cos x)^2}
 \end{aligned}$$

Question 27:

Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non-zero constants and m and n are integers):

$$\frac{x^2 \cos\left(\frac{\pi}{4}\right)}{\sin x}$$

Answer

Let $f(x) = \frac{x^2 \cos\left(\frac{\pi}{4}\right)}{\sin x}$

By quotient rule,

$$\begin{aligned}
 f'(x) &= \cos \frac{\pi}{4} \cdot \left[\frac{\sin x \frac{d}{dx}(x^2) - x^2 \frac{d}{dx}(\sin x)}{\sin^2 x} \right] \\
 &= \cos \frac{\pi}{4} \cdot \left[\frac{\sin x \cdot 2x - x^2 \cos x}{\sin^2 x} \right] \\
 &= \frac{x \cos \frac{\pi}{4} [2 \sin x - x \cos x]}{\sin^2 x}
 \end{aligned}$$

Question 28:

Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r

and s are fixed non-zero constants and m and n are integers): $\frac{x}{1 + \tan x}$

Answer

Let $f(x) = \frac{x}{1 + \tan x}$

$$f'(x) = \frac{(1 + \tan x) \frac{d}{dx}(x) - x \frac{d}{dx}(1 + \tan x)}{(1 + \tan x)^2}$$

$$f'(x) = \frac{(1 + \tan x) - x \cdot \frac{d}{dx}(1 + \tan x)}{(1 + \tan x)^2} \quad \dots \text{(i)}$$

Let $g(x) = 1 + \tan x$. Accordingly, $g(x+h) = 1 + \tan(x+h)$.

By first principle,

$$\begin{aligned}
g'(x) &= \lim_{h \rightarrow 0} \frac{g(x+h) - g(x)}{h} \\
&= \lim_{h \rightarrow 0} \left[\frac{1 + \tan(x+h) - 1 - \tan x}{h} \right] \\
&= \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{\sin(x+h)}{\cos(x+h)} - \frac{\sin x}{\cos x} \right] \\
&= \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{\sin(x+h)\cos x - \sin x \cos(x+h)}{\cos(x+h)\cos x} \right] \\
&= \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{\sin(x+h-x)}{\cos(x+h)\cos x} \right] \\
&= \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{\sin h}{\cos(x+h)\cos x} \right] \\
&= \left(\lim_{h \rightarrow 0} \frac{\sin h}{h} \right) \cdot \left(\lim_{h \rightarrow 0} \frac{1}{\cos(x+h)\cos x} \right) \\
&= 1 \times \frac{1}{\cos^2 x} = \sec^2 x \\
\Rightarrow \frac{d}{dx}(1 + \tan x) &= \sec^2 x \quad \dots \text{(ii)}
\end{aligned}$$

From (i) and (ii), we obtain

$$f'(x) = \frac{1 + \tan x - x \sec^2 x}{(1 + \tan x)^2}$$

Question 29:

Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non-zero constants and m and n are integers): $(x + \sec x)(x - \tan x)$

Answer

Let $f(x) = (x + \sec x)(x - \tan x)$

By product rule,

$$\begin{aligned}
 f'(x) &= (x + \sec x) \frac{d}{dx}(x - \tan x) + (x - \tan x) \frac{d}{dx}(x + \sec x) \\
 &= (x + \sec x) \left[\frac{d}{dx}(x) - \frac{d}{dx} \tan x \right] + (x - \tan x) \left[\frac{d}{dx}(x) + \frac{d}{dx} \sec x \right] \\
 &= (x + \sec x) \left[1 - \frac{d}{dx} \tan x \right] + (x - \tan x) \left[1 + \frac{d}{dx} \sec x \right] \quad \dots (i)
 \end{aligned}$$

Let $f_1(x) = \tan x$, $f_2(x) = \sec x$

Accordingly, $f_1(x+h) = \tan(x+h)$ and $f_2(x+h) = \sec(x+h)$

$$\begin{aligned}
 f_1'(x) &= \lim_{h \rightarrow 0} \left(\frac{f_1(x+h) - f_1(x)}{h} \right) \\
 &= \lim_{h \rightarrow 0} \left(\frac{\tan(x+h) - \tan x}{h} \right) \\
 &= \lim_{h \rightarrow 0} \left[\frac{\tan(x+h) - \tan x}{h} \right] \\
 &= \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{\sin(x+h)}{\cos(x+h)} - \frac{\sin x}{\cos x} \right] \\
 &= \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{\sin(x+h) \cos x - \sin x \cos(x+h)}{\cos(x+h) \cos x} \right] \\
 &= \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{\sin(x+h-x)}{\cos(x+h) \cos x} \right] \\
 &= \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{\sin h}{\cos(x+h) \cos x} \right] \\
 &= \left(\lim_{h \rightarrow 0} \frac{\sin h}{h} \right) \cdot \left(\lim_{h \rightarrow 0} \frac{1}{\cos(x+h) \cos x} \right) \\
 &= 1 \times \frac{1}{\cos^2 x} = \sec^2 x \\
 \Rightarrow \frac{d}{dx} \tan x &= \sec^2 x \quad \dots (ii)
 \end{aligned}$$

$$\begin{aligned}
f_2'(x) &= \lim_{h \rightarrow 0} \left(\frac{f_2(x+h) - f_2(x)}{h} \right) \\
&= \lim_{h \rightarrow 0} \left(\frac{\sec(x+h) - \sec x}{h} \right) \\
&= \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{1}{\cos(x+h)} - \frac{1}{\cos x} \right] \\
&= \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{\cos x - \cos(x+h)}{\cos(x+h)\cos x} \right] \\
&= \frac{1}{\cos x} \cdot \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{-2 \sin\left(\frac{x+x+h}{2}\right) \cdot \sin\left(\frac{x-x-h}{2}\right)}{\cos(x+h)} \right] \\
&= \frac{1}{\cos x} \cdot \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{-2 \sin\left(\frac{2x+h}{2}\right) \cdot \sin\left(\frac{-h}{2}\right)}{\cos(x+h)} \right] \\
&= \frac{1}{\cos x} \cdot \lim_{h \rightarrow 0} \left[\frac{\sin\left(\frac{2x+h}{2}\right) \left\{ \frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}} \right\}}{\cos(x+h)} \right] \\
&= \sec x \cdot \frac{\left\{ \lim_{h \rightarrow 0} \sin\left(\frac{2x+h}{2}\right) \right\} \left\{ \lim_{\frac{h}{2} \rightarrow 0} \frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}} \right\}}{\lim_{h \rightarrow 0} \cos(x+h)} \\
&= \sec x \cdot \frac{\sin x \cdot 1}{\cos x} \\
\Rightarrow \frac{d}{dx} \sec x &= \sec x \tan x \quad \dots \text{(iii)}
\end{aligned}$$

From (i), (ii), and (iii), we obtain

$$f'(x) = (x + \sec x)(1 - \sec^2 x) + (x - \tan x)(1 + \sec x \tan x)$$

Question 30:

Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r

and s are fixed non-zero constants and m and n are integers): $\frac{x}{\sin^n x}$

Answer

Let $f(x) = \frac{x}{\sin^n x}$

By quotient rule,

$$f'(x) = \frac{\sin^n x \frac{d}{dx} x - x \frac{d}{dx} \sin^n x}{\sin^{2n} x}$$

It can be easily shown that $\frac{d}{dx} \sin^n x = n \sin^{n-1} x \cos x$

Therefore,

$$\begin{aligned} f'(x) &= \frac{\sin^n x \frac{d}{dx} x - x \frac{d}{dx} \sin^n x}{\sin^{2n} x} \\ &= \frac{\sin^n x \cdot 1 - x(n \sin^{n-1} x \cos x)}{\sin^{2n} x} \\ &= \frac{\sin^{n-1} x (\sin x - nx \cos x)}{\sin^{2n} x} \\ &= \frac{\sin x - nx \cos x}{\sin^{n+1} x} \end{aligned}$$